An Extension of the Positive Real Lemma to Descriptor Systems*

Roland W. Freund
Bell Laboratories, Lucent Technologies
Room 2C–525
700 Mountain Avenue
Murray Hill, New Jersey 07974–0636, U.S.A.

Florian Jarre
Institut für Mathematik
Universität Düsseldorf
Universitätsstraße 1
D–40225 Düsseldorf, Germany

Abstract

The well-known positive real lemma characterizes positive realness of transfer functions of time-invariant linear systems via the solvability of certain linear matrix inequalities. In this paper, we propose an extension of the positive real lemma and the underlying linear matrix inequalities to descriptor systems. We show that the solvability of these linear matrix inequalities is sufficient and, under an additional assumption, also necessary for positive realness of the transfer function of time-invariant linear descriptor systems. We also study a second system of linear matrix inequalities based on a generalized Lyapunov equation. We show that the solvability of this second system is sufficient, but in general not necessary for positive realness.

Key words: linear system, descriptor system, transfer function, positive realness, linear matrix inequalities, generalized Lyapunov equation, passivity

1 Introduction

In recent years, there has been a lot of interest in reduced-order modeling of large-scale time-invariant linear dynamical systems; see, e.g., the survey paper [1] and the references given there. Much of the research in this area has been motivated by the need for efficient reduced-order modeling techniques in the simulation of VLSI circuits. In particular, such techniques are often applied to passive systems, and then it is crucial that the reduced-order model preserves the passivity of the original system. Unfortunately, reduced-order models that are in a Padé sense optimal approximations to the original system are not passive in general. However, these models are often "almost" passive, and then the question arises of how to modify the models so that they become passive, without giving up too much of the optimal approximation property. For single-input single-output systems, such an approach is discussed in [2].

*Numerical Analysis Manuscript No. 00–3–09, Bell Laboratories, Murray Hill, New Jersey, December 2000. Available on WWW at `http://cm.bell-labs.com/cs/doc/00`.

For time-invariant linear dynamical systems, passivity is equivalent to positive realness of the system's transfer function; see, e.g., [3]. Hence the passivity of a given reduced-order model can be checked by determining if the associated transfer function is positive real. For single-input single-output systems, the transfer function is scalar valued, and testing if it is positive real can be done by means of relatively elementary means; see, e.g., [4] and the references given there. However, in the general case of multi-input multi-output systems, determining if the associated matrix-valued transfer function is positive real is a much harder problem. For regular systems, one possible approach is to employ the celebrated positive real lemma, which characterizes positive realness of matrix-valued transfer functions by means of the solvability of certain linear matrix inequalities. The positive real lemma was developed in the 1960s; see [5, 6, 7, 8, 9, 10]. Various implications and applications of the positive real lemma are described in [3, 11, 12].

The classical positive real lemma is restricted to regular time-invariant linear systems and cannot be applied directly to singular (so-called *descriptor*) systems. However, in many interesting applications, in particular in circuit simulation, the given system is in fact singular. The purpose of this paper is to develop extensions of the positive real lemma and the underlying linear matrix inequalities to descriptor systems. In particular, we show that the solvability of certain linear matrix inequalities is sufficient and, under an additional assumption, also necessary for positive realness of the transfer function of time-invariant linear descriptor systems. We also study a second system of linear matrix inequalities based on a generalized Lyapunov equation. We show that the solvability of this second system is sufficient, but in general not necessary for positive realness.

The remainder of this paper is organized as follows. In Section 2, we formulate the problem to be addressed in this paper and also review the positive real lemma for regular systems. Our main results are stated in Sections 3 and 4. Most of the proofs are contained in Section 5. Some known results, which are used in these proofs, are stated in Section 6. Finally, in Section 7, we make some concluding remarks.

Throughout this paper, we use the following notation. For a real square matrix P, we write $P \geq 0$ if $P = P^T$ is symmetric and if P is positive semidefinite, i.e., $x^T P x \geq 0$ for all real vectors x of suitable dimension. For a complex number α or a complex matrix P, we denote the complex conjugate by $\overline{\alpha}$ or $\overline{P}$. The $n \times n$ identity matrix is denoted by I_n and the zero matrix by 0. If the dimension of I_n is apparent from the context, we simply write $I = I_n$. The actual dimension of 0 will always be apparent from the context. The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ is the real part of s.

2 The Problem

We consider m-input m-output time-invariant linear dynamical systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^T x(t) + Du(t). \end{aligned} \tag{1}$$

Here, $A, E \in \mathbb{R}^{n \times n}$, $B, C \in \mathbb{R}^{n \times m}$, and $D \in \mathbb{R}^{m \times m}$ are given matrices, $x(t)$ is the vector of state variables, $u(t)$ is the vector of inputs, and $y(t)$ the vector of outputs. The linear system (1) is called *regular* if the matrix E in (1) is nonsingular, and it is called *singular* or a *descriptor system* if E is singular. Note that, in the regular case, the linear system (1) can always be re-written as

$$\dot{x}(t) = \left(E^{-1}A\right)x(t) + \left(E^{-1}B\right)u(t),$$
$$y(t) = C^T x(t) + Du(t),$$

which is just a system (1) with $E = I$. Throughout this paper, the only assumption on the matrices A, $E \in \mathbb{R}^{n\times n}$ in (1) is that the matrix pencil $A - sE$ is *regular*, i.e., the matrix $A - sE$ is singular for only finitely many values of $s \in \mathbb{C}$.

The system (1) can be described by its *transfer function*

$$H(s) := D + C^T\left(sE - A\right)^{-1}B, \quad s \in \mathbb{C}. \tag{2}$$

Note that

$$H : \mathbb{C} \mapsto \left(\mathbb{C} \cup \infty\right)^{m\times m}, \tag{3}$$

is an $m \times m$-matrix-valued rational function. Conversely, given a rational function (3), H, any representation of H of the form (2) is called a *realization* of H. A realization (2) of H is said to be *minimal* if the dimension n of the matrices A and E in (2) is as small as possible. In general, we do not assume that the realization (2) of H is minimal.

It is well known that a system described by (1) is *passive*, i.e., it does not generate energy, if, and only if, its transfer function (2) is *positive real*; see, e.g., [3]. A precise definition of positive realness is as follows.

Definition 1 *An $m \times m$-matrix-valued function $H : \mathbb{C} \mapsto \left(\mathbb{C} \cup \infty\right)^{m\times m}$ is called positive real if the following three conditions are satisfied:*

(i) *H is analytic in $\mathbb{C}_+ := \{\, s \in \mathbb{C} \mid \operatorname{Re} s > 0 \,\}$;*

(ii) *$H(\overline{s}) = \overline{H(s)}$ for all $s \in \mathbb{C}$;*

(iii) *$H(s) + \left(H(s)\right)^H \geq 0$ for all $s \in \mathbb{C}_+$.*

Note that, for functions of the form (2), condition (ii) of Definition 1 is always satisfied since the matrices A, E, B, C, and D in (2) are assumed to be real.

The classical positive real lemma characterizes positive realness of transfer functions of regular linear systems (1) via the solvability of certain linear matrix inequalities (LMIs); see, e.g., [10], [3, Chapter 5], [13, Section 13.5]. The result can be summarized as follows.

Theorem A (Positive real lemma for regular systems [10])
Let H be a real $m \times m$-matrix-valued rational function of the form

$$H(s) = D + C^T\left(sI - A\right)^{-1}B. \tag{4}$$

a) (Sufficient condition)
If the LMIs

$$\begin{bmatrix} A^TX + XA & XB - C \\ B^TX - C^T & -D - D^T \end{bmatrix} \leq 0 \quad \text{and} \quad X \geq 0 \tag{5}$$

have a solution X, then H is positive real.

b) (Necessary condition)
Suppose that (4) is a minimal realization of H. If H is positive real, then there exists a solution X of the LMIs (5).

The purpose of this paper is to extend the result of Theorem A to singular linear systems (1).

3 A Positive Real Lemma for Descriptor Systems

It turns out that the positive realness of functions H of the form (2) can be characterized by means of the following LMIs:

$$\begin{bmatrix} A^T X + X^T A & X^T B - C \\ B^T X - C^T & -D - D^T \end{bmatrix} \leq 0 \quad \text{and} \quad E^T X = X^T E \geq 0. \tag{6}$$

Note that, in view of Proposition 5 (stated in Section 5.5 below), finding a solution X of the LMIs (6) is equivalent to finding matrices X, $L \in \mathbb{R}^{n \times n}$, and $W \in \mathbb{R}^{n \times m}$ such that

$$\begin{aligned} A^T X + X^T A &= -LL^T, \\ X^T B - C &= -LW, \\ D + D^T &\geq W^T W, \\ E^T X = X^T E &\geq 0. \end{aligned} \tag{7}$$

First, we show that the solvability of (6) is indeed sufficient for the positive realness of H.

Theorem 1 (Sufficient condition)
If (6) *has a solution* X*, then the function* (2), H*, is positive real.*

Proof Let $s \in \mathbb{C}_+$ be any point such that s is not a pole of H. In particular, for such an s, the matrix $A - sE$ is nonsingular. Let X, L, and W be solutions of the equivalent formulation (7) of the LMIs (6). The first and last relation in (7) imply that

$$\begin{aligned} (sE - A)^H X + X^T (sE - A) &= -A^T X - X^T A + \overline{s} E^T X + s X^T E \\ &= LL^T + (\operatorname{Re} s) \left(E^T X + X^T E\right) - i \left(\operatorname{Im} s\right)(E^T X - X^T E) \\ &= LL^T + 2(\operatorname{Re} s) E^T X. \end{aligned} \tag{8}$$

Next, note that

$$(sE - A) F = B, \quad \text{where} \quad F = F(s) := (sE - A)^{-1} B. \tag{9}$$

By (2), (9), and the second relation in (7), we have

$$\begin{aligned} H(s) &= D + C^T F \\ &= D + W^T L^T F + B^T X F \\ &= D + W^T L^T F + F^H (sE - A)^H X F. \end{aligned} \tag{10}$$

Using (10), (8), and the third relation in (7), we get

$$\begin{aligned} H(s) + (H(s))^H &= D + D^T + W^T L^T F + F^H L W + F^H [(sE - A)^H X + X^T (sE - A)] F \\ &\geq W^T W + W^T L^T F + F^H L W + F^H [(LL^T + 2(\operatorname{Re} s) E^T X] F \\ &= \left(W + L^T F\right)^H \left(W + L^T F\right) + 2(\operatorname{Re} s) F^H \left(E^T X\right) F. \end{aligned} \tag{11}$$

Obviously,

$$\left(W + L^T F\right)^H \left(W + L^T F\right) \geq 0.$$

Moreover, since $\operatorname{Re} s > 0$ and $E^T X = \left(E^T X\right)^T \geq 0$, we have

$$(\operatorname{Re} s) F^H \left(E^T X\right) F \geq 0.$$

Thus, by (11),

$$H(s) + (H(s))^H \geq 0. \tag{12}$$

Recall that s was assumed to be any $s \in \mathbb{C}_+$ such that s is not a pole of H. Now suppose H has a pole $s_0 \in \mathbb{C}_+$. Then, there exists a pointed neighborhood of s_0 that is free of any poles of H, and thus, H satisfies (12) in that pointed neighborhood of s_0. However, this is impossible if s_0 were indeed a pole of H. Therefore, H does not have any poles in $\mathbb{C}_+$, and (12) holds true for all $s \in \mathbb{C}_+$. Hence H is positive real, and the proof is complete. □

Recall from Theorem A that for transfer functions (4), H, of regular systems, the solvability of the LMIs (5) is also necessary for positive realness of H, provided that (4) is a minimal realization of H. In the descriptor case, we need an additional condition in order to show that solvability of the LMIs (6) is also necessary for positive realness. To formulate this condition, we first note that, since H is rational, it has an expansion about $s = \infty$ of the form

$$H(s) = \sum_{i=-\infty}^{p} M_i s^i. \tag{13}$$

Here, $p := 0$ if $s = \infty$ is not a pole of H, and $p \geq 1$ is the order of the pole $s = \infty$ otherwise. The additional condition for the necessity of the solvability of the LMIs (6) is as follows:

$$D + D^T \geq M_0 + M_0^T. \tag{14}$$

Here, M_0 is the constant term in the expansion (13). We remark that in the regular case, the condition (14) is trivially satisfied. Indeed, in view of (4) and (13), we have $p = 0$ and $D = H(\infty) = M_0$ in the regular case.

Our result on the necessity of the solvability of the LMIs (6) can now be stated as follows; the proof of this theorem is given in Section 5.3.

Theorem 2 (Necessary condition)
Assume that the representation (2) *of H is a minimal realization of H and that the matrix M_0 in the expansion* (13) *of H about $s = \infty$ satisfies condition* (14)*. Then, if H is positive real, there exists a solution X of the LMIs* (6)*.*

4 Another Sufficient Condition

In this section, we consider yet another system of LMIs and show that its solvability is sufficient, but not necessary in general, for positive realness.

The system of LMIs is as follows:

$$\begin{aligned} A^T \hat{X} E + E^T \hat{X}^T A &= -LL^T, \\ E^T \hat{X}^T B - C &= -LW, \\ D + D^T &\geq W^T W, \\ E^T \hat{X} E = E^T \hat{X}^T E &\geq 0. \end{aligned} \tag{15}$$

Here, the unknowns are the matrices $\hat{X}$, $L \in \mathbb{R}^{n \times n}$, and $W \in \mathbb{R}^{n \times m}$.

Theorem 3 (Sufficient condition)
If (15) *has a solution, the function* (2), H, *is positive real.*

Proof Let $\hat{X}$, L, and W be a solution of (15). Then, $X := \hat{X}E$, L, and W is a solution of (7), and by applying Theorem 1, it follows that H is positive real. □

Note that the statement of Theorem 1 is stronger than that of Theorem 3; the matrix X in (7) may have full rank even when E is singular, while the rank of $X = \hat{X}E$ is less or equal to the rank of E.

We obtain an even simpler, but stronger, condition by replacing the last relation in (15) as follows:

$$\begin{aligned} A^T\hat{X}E + E^T\hat{X}^TA &= -LL^T, \\ E^T\hat{X}^TB - C &= -LW, \\ D + D^T &\geq W^TW, \\ \hat{X} = \hat{X}^T &\geq 0. \end{aligned} \tag{16}$$

Obviously, (15) and (16) are equivalent if E is nonsingular. Moreover, in this case, also (7) and (15) are equivalent. Indeed, assume that E is nonsingular, and let X be any solution of (7). Set $\hat{X} := XE^{-1}$. By multiplying the last relation in (7) from the left by E^{-T} and from the right by E^{-1}, we get

$$\begin{aligned} \hat{X} &= XE^{-1} = E^{-T}X^T = (XE^{-1})^T = \hat{X}^T, \\ \hat{X} &= E^{-T}(E^TX)E^{-1} \geq 0. \end{aligned}$$

Since $X = \hat{X}E$ and $X^T = E^T\hat{X}$, the first two relations of (7) and (15) are also equivalent.

However, as the following proposition states, (15) and (16) do not have a solution in general, even if H is positive real. A proof of this proposition is given in Section 5.4.

Proposition 1 *Assume that the representation* (2) *of* H *is a minimal realization of* H *and that the matrix* E *is singular. Then, the LMIs* (15) *and* (16) *do not have any solution.*

The LMIs (7), (15), and (16) can all be viewed as natural extensions of the LMIs underlying the positive real lemma for regular systems. The LMIs (16) are numerically attractive since they involve a symmetric matrix $\hat{X}$. In contrast, the matrix X in (7) is nonsymmetric in general. A numerical comparison of the two approaches based on (7) and (16) for the case of nearly singular matrices E is subject of the forthcoming paper [14].

5 Proofs

In this section, we present proofs of Theorem 2 and of Proposition 1.

5.1 Reduction to Weierstrass Form

To simplify the proof of Theorem 2, we first show that the data matrices A and E can be assumed to be in Weierstrass form with a nilpotent matrix of special form.

Recall that A and E are real $n \times n$ matrices and that $A - sE$ is assumed to be a regular pencil. For any such pair of matrices A and E, there exist real nonsingular matrices P and Q such that

$$QAP = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix}, \quad QEP = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}, \tag{17}$$

and the $(n-q) \times (n-q)$ submatrix N is nilpotent, i.e., $N^l = 0$ for some integer $l \geq 1$. The matrices (17) are called the Weierstrass form of the pencil $A - sE$; see, e.g., [15, Chapter XII] or [16, Section 1.7]. Note that the Weierstrass form readily follows from the more general Kronecker normal form for rectangular matrix pencils. However, we stress that the submatrix A_1 in (17) is not assumed to be in Jordan normal form.

Next, we re-write the transfer function (2), H, as follows:

$$\begin{aligned} H(s) &= D + C^T \left(Q^{-1}\left(sQEP - QAP\right)P^{-1}\right)^{-1} B \\ &= D + \left(P^T C\right)^T \left(sQEP - QAP\right)^{-1} (QB). \end{aligned} \tag{18}$$

Similarly, using the fact that nonsingular congruence transformations do not change the signature of a matrix, the LMIs (7) are equivalent to the following relations:

$$\begin{aligned} (QAP)^T Q^{-T} XP + \left(Q^{-T}XP\right)^T QAP &= -P^T L L^T P, \\ \left(Q^{-T}XP\right)^T QB - P^T C &= -P^T L W, \\ D + D^T &\geq W^T W, \\ (QEP)^T Q^{-T} XP = \left(Q^{-T}XP\right)^T QEP &\geq 0. \end{aligned} \tag{19}$$

Observe that relations (2) and (7) coincide with (18) and (19) when A, E, B, C, L, and X are replaced by QAP, QEP, QB, $P^T C$, $P^T L$, and $Q^{-T} XP$, respectively. Hence, without loss of generality, we may assume that the matrices A and E are already in Weierstrass form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}. \tag{20}$$

It follows from (20) that

$$(sE - A)^{-1} = \begin{bmatrix} (sI - A_1)^{-1} & 0 \\ 0 & (sN - I)^{-1} \end{bmatrix}. \tag{21}$$

Moreover, since N is nilpotent with $N^l = 0$, we have

$$(sN - I)^{-1} = -\sum_{i=0}^{\infty} N^i s^i = -\sum_{i=0}^{l-1} N^i s^i. \tag{22}$$

Inserting (22) into (21) and the result into the representation (2) of H, we get

$$H(s) = D + C_1^T (sI - A_1)^{-1} B_1 - C_2^T \left(\sum_{i=0}^{l-1} N^i s^i\right) B_2. \tag{23}$$

Here, B_1, B_2, C_1, and C_2 are defined by partitionings

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

of B and C that conform with the partitioning (20) of A and E.

The next proposition shows that for positive real functions H, the nilpotent matrix N in (20) can be assumed to be of a very special form.

Proposition 2 *Assume that H is a positive real function and that the representation* (2) *of H is a minimal realization, with matrices A and E of the form* (20) *and N is nilpotent. Then, N is either the empty matrix or its Jordan canonical form consists only of Jordan blocks of size* 2.

Proof Assume that H is positive real and is given by a minimal realization with matrices A and E of the form (20). Moreover, without loss of generality, we can assume that the nilpotent matrix N in (20) is in Jordan canonical form.

By condition (v) of Theorem D in Section 6, N cannot have any 1×1 Jordan blocks. So either N is the empty matrix, or N consists only of Jordan blocks of size 2 or larger. In the latter case, we need to show that all Jordan blocks are of size 2. Suppose that N has Jordan blocks of size larger than 2. Let $j_0 > 2$ be the maximum size of such a block and let n_0 be the number of blocks of size j_0. Let B_{last} denote the submatrix of B_2 consisting of all rows of B_2 corresponding to the last row of each of the n_0 Jordan blocks of size j_0. Similarly, let C_{first} denote the submatrix of C_2 consisting of all rows of C_2 corresponding to the first row of each of the n_0 Jordan blocks of size j_0. Note that both B_{last} and C_{first} are $n_0 \times m$ matrices. By construction, we have

$$C_2^T N^{j_0-1} B_2 = C_{\text{first}}^T B_{\text{last}}.$$

Moreover, the conditions (ii) and (iv) of Theorem D in Section 6 imply that both B_{last} and C_{first} have full row rank n_0. Therefore,

$$C_2^T N^{j_0-1} B_2 = C_{\text{first}}^T B_{\text{last}} \neq 0.$$

It follows that in the representation (23) of H there is a term

$$C_2^T N^p B_2 \neq 0, \quad \text{where} \quad p := j_0 - 1 \geq 2.$$

and thus $s = \infty$ is a pole of H of order at least 2. However, this contradicts the assumption that H is positive real. Indeed, by Theorem B in Section 6, a positive real function cannot have a pole of order $p \geq 2$ at $s = \infty$. Therefore, we have shown that either N is the empty matrix or N consists only of Jordan blocks of size 2. □

Under the assumptions of Proposition 2, we can thus assume that the matrix N in (20) is either empty, i.e., $n = q$ in (20), or N consists of only 2×2 Jordan blocks. In the latter case, we can further permute the rows and columns of N such that N is of the form

$$N = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \tag{24}$$

where all 4 blocks are square and of the same size.

5.2 Solution of Certain Nonsymmetric LMIs

In this section, we present a parametrization of all solutions X of the LMIs

$$\begin{aligned} A^T X + X^T A &\leq 0, \\ E^T X = X^T E &\geq 0. \end{aligned} \tag{25}$$

Here, we assume that A and E are of the form (20) with N of the form (24).

We partition X in (25) in accordance with (20) as follows:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}. \tag{26}$$

Inserting (20) and (26) into (25), we get

$$\begin{bmatrix} A_1^T X_1 + X_1^T A_1 & A_1^T X_2 + X_3^T \\ X_2^T A_1 + X_3 & X_4 + X_4^T \end{bmatrix} \leq 0, \tag{27}$$

$$\begin{bmatrix} X_1 & X_2 \\ N^T X_3 & N^T X_4 \end{bmatrix} = \begin{bmatrix} X_1^T & X_3^T N \\ X_2^T & X_4^T N \end{bmatrix} \geq 0. \tag{28}$$

In particular, (27) and (28) imply that the subblock X_4 satisfies

$$\begin{aligned} X_4 + X_4^T &\leq 0, \\ N^T X_4 = X_4^T N &\geq 0. \end{aligned} \tag{29}$$

Next, let

$$X_4 = \begin{bmatrix} Z_0 & Z_1 \\ Z_3 & Z_2 \end{bmatrix} \tag{30}$$

be a partitioning conforming with the partitioning (24) of N. Inserting (30) into (29), we get

$$\begin{bmatrix} Z_0 + Z_0^T & Z_1 + Z_3^T \\ Z_1^T + Z_3 & Z_2 + Z_2^T \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} 0 & 0 \\ Z_0 & Z_1 \end{bmatrix} = \begin{bmatrix} 0 & Z_0^T \\ 0 & Z_1^T \end{bmatrix} \geq 0.$$

These relations are satisfied if, and only if, $Z_0 = 0$, $Z_1 = Z_1^T \geq 0$, $Z_3 = -Z_1^T$, and $Z_2 + Z_2^T \leq 0$. Thus, we have shown that for any solution X of (25), the subblock X_4 must be of the form

$$X_4 = \begin{bmatrix} 0 & Z_1 \\ -Z_1 & Z_2 \end{bmatrix}, \quad \text{where} \quad Z_1 \geq 0 \quad \text{and} \quad Z_2 + Z_2^T \leq 0. \tag{31}$$

Moreover, note that for any X_4 of the form (31),

$$X_4 + X_4^T = \begin{bmatrix} 0 & 0 \\ 0 & Z_2 + Z_2^T \end{bmatrix} \leq 0 \quad \text{and} \quad N^T X_4 = X_4^T N = \begin{bmatrix} 0 & 0 \\ 0 & Z_1 \end{bmatrix} \geq 0. \tag{32}$$

The first relation in (32) implies that the block $X_2^T A_1 + X_3$ in (27) must be of the form

$$X_2^T A_1 + X_3 = \begin{bmatrix} 0 & 0 \\ \star & \star \end{bmatrix}. \tag{33}$$

The second relation in (32) implies that the blocks X_2^T and X_3 in (28) must be of the form

$$X_2^T = N^T X_3 = \begin{bmatrix} 0 & 0 \\ Y_3 & Y_4 \end{bmatrix} \quad \text{and} \quad X_3 = \begin{bmatrix} Y_3 & Y_4 \\ Y_1 & Y_2 \end{bmatrix}. \tag{34}$$

Inserting X_2^T and X_3 from (34) into (33) shows that $Y_3 = 0$ and $Y_4 = 0$. Thus, by (34), we have that

$$X_2 = 0 \quad \text{and} \quad X_3 = \begin{bmatrix} 0 & 0 \\ Y_1 & Y_2 \end{bmatrix}. \tag{35}$$

Note that, by (28), $X_1 \geq 0$. Furthermore, for any $X_1 \geq 0$, any X_2, X_3 of the form (35), and any X_4 of the form (31), the relation (27) reduces to

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{$A_1^T X_1 + X_1 A_1$}} & 0 & Y_1^T \\ & & 0 & Y_2^T \\ \hline 0 & 0 & 0 & 0 \\ Y_1 & Y_2 & 0 & Z_2 + Z_2^T \end{array}\right] \leq 0.$$

In summary, we have proved the following result.

Proposition 3 *Let A and E be matrices of the form* (20) *with N of the form* (24)*. Then all solutions X of* (25) *are of the form*

$$X = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{X_1}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & Z_1 \\ Y_1 & Y_2 & -Z_1 & Z_2 \end{array}\right].$$

Moreover, for all such X, the matrix equations (25) *are equivalent to the conditions*

$$\left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$A_1^T X_1 + X_1 A_1$}} & Y_1^T \\ & & Y_2^T \\ \hline Y_1 & Y_2 & Z_2 + Z_2^T \end{array}\right] \leq 0,$$

$$X_1 \geq 0, \quad Z_1 \geq 0.$$

5.3 Proof of Theorem 2

In this section, we give a detailed proof of Theorem 2. As shown in Section 5.1, we may assume without loss of generality, that A and E are of the form (20) with N of the form (24).

Let H be a positive real function, and let (2) be a minimal realization of H. First, we rewrite H as follows:

$$\begin{aligned} H(s) &= D + C^T (sE - A)^{-1} B \\ &= D + C_1^T (sI - A_1)^{-1} B_1 + C_2^T (sN - I)^{-1} \\ &= D - C_2^T B_2 + C_1^T (sI - A_1)^{-1} B_1 - sC_2^T N B_2. \end{aligned} \tag{36}$$

Since H is assumed to be positive real, it satisfies the conditions for positive realness given in Theorem B in Section 6 below.

Consider the matrix $M_1 := -C_2^T N B_2$ in (36). If $M_1 = 0$, then $s = \infty$ is not a pole of H. Otherwise, $s = \infty$ is a pole of H, and by Theorem B, it follows that

$$M_1 = -C_2^T N B_2 \geq 0. \tag{37}$$

Next, set
$$H_1(s) := D - C_2^T B_2 + C_1^T (sI - A_1)^{-1} B_1. \tag{38}$$
Then, by (36) and (37), we have
$$H(s) = H_1(s) + sM_1, \quad \text{where} \quad M_1 \geq 0.$$
In particular, this implies
$$H(i\omega) + (H(i\omega))^H = H_1(i\omega) + (H_1(i\omega))^H$$
for all $\omega \in \mathbb{R}$, with $i\omega$ not a pole of H. Thus, H_1 also satisfies the conditions given in Theorem B, and hence H_1 is also positive real.

By comparing (36) and (13), we deduce that
$$M_0 = D - C_2^T B_2. \tag{39}$$
By (39), the condition (14), $D + D^T \geq M_0 + M_0^T$, is equivalent to
$$C_2^T B_2 + B_2^T C_2 \geq 0. \tag{40}$$

By Theorem D, the assumption that the representation (2) is a minimal realization of H implies that (38) is a minimal realization of H_1. Moreover, for later use we note that, by Theorem D,
$$\begin{bmatrix} N & B_2 \end{bmatrix} \quad \text{has full rank.} \tag{41}$$

By applying the standard positive real lemma (Theorem A) to H_1, it follows that there exist matrices X_1, L_1, and W_1 such that
$$\begin{aligned} A_1^T X_1 + X_1 A_1 &= -L_1 L_1^T, \\ X_1 B_1 - C_1 &= -L_1 W_1, \\ M_0 + M_0^T &\geq W_1^T W_1, \\ X_1 &\geq 0. \end{aligned} \tag{42}$$

Next, we construct a second set of matrices Z_1, Z_2, L_2, and W_2 that together with X_1, L_1, and W_1 then yield a solution of the LMIs (7).

Proposition 4 *Suppose that the matrices B_2 and C_2 satisfy* (37), (40), *and* (41). *Then, there exist real matrices Z_1, Z_2, L_2, and W_2 such that*
$$\begin{aligned} \begin{bmatrix} 0 & -Z_1 \\ Z_1 & Z_2^T \end{bmatrix} B_2 - C_2 &= -\begin{bmatrix} 0 \\ L_2 W_2 \end{bmatrix}, \\ Z_2 + Z_2^T &= -L_2 L_2^T, \\ Z_1 &\geq 0. \end{aligned} \tag{43}$$
Moreover, W_2 satisfies the equation
$$W_2^T W_2 = C_2^T B_2 + B_2^T C_2. \tag{44}$$

Proof First, we partition B_2 so that

$$B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \quad \text{and} \quad [\, N \quad B_2 \,] = \begin{bmatrix} 0 & I & B_{21} \\ 0 & 0 & B_{22} \end{bmatrix}.$$

The condition (41) implies that the submatrix B_{22} has full row rank. Hence there exists a nonsingular matrix T such that

$$B_2 T = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} T = \begin{bmatrix} \beta_1 & \beta_2 \\ I & 0 \end{bmatrix}. \tag{45}$$

Let

$$\begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{bmatrix} := C_2 T \tag{46}$$

be the corresponding partitioning of $C_2 T$. By multiplying (37) from the left and right by T^T and T, respectively, and by using (24), (45), and (46), it follows that

$$0 \leq -\,(C_2 T)^T \, N \, (B_2 T) = -\,[\, \gamma_1 \quad \gamma_2 \,]^T \, [\, I \quad 0 \,].$$

This implies that

$$Z_1 := -\gamma_1 \geq 0 \quad \text{and} \quad \gamma_2 = 0. \tag{47}$$

Similarly, from (40), (45), (46), and (47), it follows that

$$0 \leq T^T \left(C_2^T B_2 + B_2^T C_2 \right) T = \begin{bmatrix} -Z_2 - Z_2^T & -Z_1\beta_2 + \gamma_4 \\ (-Z_1\beta_2 + \gamma_4)^T & 0 \end{bmatrix}, \tag{48}$$

where

$$Z_2 := Z_1 \beta_1 - \gamma_3. \tag{49}$$

We further deduce from (48) that

$$-Z_2 - Z_2^T \geq 0 \quad \text{and} \quad \gamma_4 = Z_1 \beta_2. \tag{50}$$

Finally, let L_2 be a Cholesky factor of

$$-Z_2 - Z_2^T = L_2 L_2^T, \tag{51}$$

and set

$$W_2 := [\, L_2^T \quad 0 \,] \, T^{-1}. \tag{52}$$

Note that, by construction, W_2 satisfies the relation (44).

It remains to show that (43) is fulfilled. By (47) and (51), Z_1 and Z_2 satisfies the second and third relation in (43). Furthermore, using (45)–(52), it is straightforward to verify that

$$\begin{aligned}
\begin{bmatrix} 0 & -Z_1 \\ Z_1 & Z_2^T \end{bmatrix} B_2 - C_2 &= \left(\begin{bmatrix} 0 & -Z_1 \\ Z_1 & Z_2^T \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 \\ I & 0 \end{bmatrix} - \begin{bmatrix} -Z_1 & 0 \\ \gamma_3 & Z_1\beta_2 \end{bmatrix} \right) T^{-1} \\
&= \begin{bmatrix} 0 & 0 \\ Z_2 + Z_2^T & 0 \end{bmatrix} T^{-1} \\
&= -\begin{bmatrix} 0 \\ L_2 W_2 \end{bmatrix}.
\end{aligned}$$

Thus the first relation in (43) also holds true, and the proof is complete. □

Now let X_1, L_1, and W_1 be the matrices satisfying (42), let Z_1, Z_2, L_2, and W_2 be the matrices satisfying (43), and set

$$X = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{X_1}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & Z_1 \\ 0 & 0 & -Z_1 & Z_2 \end{array}\right], \quad L = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{L_1}} & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_2 \end{array}\right], \quad \text{and} \quad W = \left[\begin{array}{c} W_1 \\ \hline 0 \\ W_2 \end{array}\right].$$

Then using (42)–(44) and Proposition 3, it is straightforward to verify that X, L, and W fulfill the relations

$$\begin{aligned} A^T X + X^T A &= -LL^T, \\ X^T B - C &= -LW, \\ D + D^T &\geq W^T W, \\ E^T X = X^T E &\geq 0. \end{aligned}$$

Thus we have proved Theorem 2.

5.4 Proof of Proposition 1

In this section, we give a proof of Proposition 1.

Observe that any matrix $\hat{X}$ satisfying (16) also satisfies (15). Hence it suffices to show that the LMIs (15) cannot have a solution.

First, we give a characterization of all solutions $\hat{X}$ of the LMIs

$$\begin{aligned} A^T \hat{X} E + E^T \hat{X}^T A &\leq 0, \\ E^T \hat{X} E = E^T \hat{X}^T E &\geq 0. \end{aligned} \tag{53}$$

Here, the matrices A and E are assumed to be of the form (20) with N of the form (24). We partition $\hat{X}$ in (53) in accordance with (20) as follows:

$$\hat{X} = \begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_3 & \hat{X}_4 \end{bmatrix}. \tag{54}$$

Inserting (20) and (54) into (53), we get

$$\begin{bmatrix} A_1^T \hat{X}_1 + \hat{X}_1 A_1 & A_1^T \hat{X}_2 N + \hat{X}_3^T \\ (A_1^T \hat{X}_2 N + \hat{X}_3^T)^T & \hat{X}_4 N + N^T \hat{X}_4^T \end{bmatrix} \leq 0, \tag{55}$$

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 N \\ N^T \hat{X}_3 & N^T \hat{X}_4 N \end{bmatrix} = \begin{bmatrix} \hat{X}_1^T & \hat{X}_3^T N \\ N^T \hat{X}_2^T & N^T \hat{X}_4^T N \end{bmatrix} \geq 0. \tag{56}$$

In particular, (55) implies that the subblock $\hat{X}_4$ satisfies

$$\hat{X}_4 N + N^T \hat{X}_4^T \leq 0. \tag{57}$$

Next, let

$$\hat{X}_4 = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \tag{58}$$

be a partitioning of $\hat{X}_4$ conforming with the partitioning (24) of N. Then, relation (57) can be stated as follows:

$$\begin{bmatrix} 0 & Z_1 \\ Z_1^T & Z_3 + Z_3^T \end{bmatrix} \leq 0.$$

This implies $Z_1 = 0$, and together with (24) and (58), it follows that $N^T \hat{X}_4 N = 0$. Inserting this relation into (56), we further conclude that $\hat{X}_2 N = 0$ and $N^T \hat{X}_3 = 0$. In view of (24), this means that $\hat{X}_2$ and $\hat{X}_3$ are of the form

$$\hat{X}_2 = [\,0 \quad U\,] \quad \text{and} \quad \hat{X}_3 = \begin{bmatrix} 0 \\ V^T \end{bmatrix}, \tag{59}$$

where the submatrices 0, U, and V are all of the same size.

Inserting (59) and (58) (where $Z_1 = 0$) into (54), it follows that $\hat{X}$ is of the form

$$\hat{X} = \left[\begin{array}{c|cc} \hat{X}_1 & 0 & U \\ \hline 0 & 0 & Z_2 \\ V^T & Z_3 & Z_4 \end{array}\right], \tag{60}$$

where, by (56), $\hat{X}_1 = \hat{X}_1^T \geq 0$. Furthermore, by (55), (60), and (24), we have

$$A^T \hat{X} E + E^T \hat{X}^T A = \left[\begin{array}{c|cc} A_1^T \hat{X}_1 + \hat{X}_1^T A_1 & 0 & V \\ \hline 0 & 0 & 0 \\ V^T & 0 & Z_3 + Z_3^T \end{array}\right] \leq 0. \tag{61}$$

Next, we show that the system

$$\begin{aligned} A^T \hat{X} E + E^T \hat{X}^T A &= -LL^T, \\ E^T \hat{X}^T B - C &= -LW, \\ E^T \hat{X} E = E^T \hat{X}^T E &\geq 0 \end{aligned} \tag{62}$$

cannot have a solution when (2) is a minimal realization of H. As we have just shown, all $\hat{X}$ satisfying the first and third relation in (62) are of the form (60). Moreover, by (61), it follows that L in (62) must be of the form

$$L = \left[\begin{array}{c} L_1 \\ \hline 0 \\ L_2 \end{array}\right],$$

where the partitioning of L is conforming with (61). We partition B and C accordingly:

$$B = \left[\begin{array}{c} B_1 \\ \hline B_{21} \\ B_{22} \end{array}\right] \quad \text{and} \quad C = \left[\begin{array}{c} C_1 \\ \hline C_{21} \\ C_{22} \end{array}\right].$$

Then, the second relation in (62) is equivalent to

$$E^T \hat{X}^T B - C = \left[\begin{array}{c} \hat{X}_1 B_1 + V B_{22} \\ \hline 0 \\ Z_3^T B_{22} \end{array}\right] - \left[\begin{array}{c} C_1 \\ \hline C_{21} \\ C_{22} \end{array}\right] = \left[\begin{array}{c} -L_1 W \\ \hline 0 \\ -L_2 W \end{array}\right].$$

The second block row of this equation implies that $C_{21} = 0$. Now recall that (2) is assumed to be a minimal realization of H. Thus, in view of condition (iv) of Theorem D, the matrix

$$[\, N^T \quad C_2 \,] = \begin{bmatrix} 0 & 0 & C_{21} \\ I & 0 & C_{22} \end{bmatrix}$$

must have full row rank. Hence $C_{21} = 0$ would only be possible if N is the empty matrix, which in turn implies that $E = I$. Thus, in the descriptor case, i.e., if E is singular, and if (2) is a minimal realization of H, then (62) cannot have a solution. Since the relations (62) are all included in (16), this means that the LMIs (16) do not have a solution.

5.5 Positive Semidefiniteness of a Block Matrix

The following proposition gives a characterization of the positive semidefiniteness of a block matrix in terms of certain conditions on its subblocks.

Proposition 5 *Let*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$$

be a real symmetric matrix. Then $M \geq 0$ if, and only if, there exist real matrices L and W such that

$$\begin{aligned} M_{11} &= LL^T, \\ M_{12} &= LW, \\ M_{22} &\geq W^T W. \end{aligned} \tag{63}$$

Proof Let L and W satisfy (63). Then

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \geq \begin{bmatrix} LL^T & LW \\ W^T L^T & W^T W \end{bmatrix} = \begin{bmatrix} L \\ W^T \end{bmatrix} [\, L^T \quad W \,] \geq 0.$$

Conversely, assume that $M \geq 0$. Let T be any real square matrix such that $M = T^T T$. Let $T = QR$ be the QR factorization of T with an orthogonal matrix Q and an upper triangular matrix R. Then, $M = T^T T = R^T R$ is a Cholesky factorization of M. We partition the matrix R in the form

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} =: \begin{bmatrix} L^T & W \\ 0 & R_{22} \end{bmatrix}.$$

From $M = R^T R$ follows that $M_{11} = LL^T$, $M_{12} = LW$, and $M_{22} = W^T W + R_{22}^T R_{22} \geq W^T W$. Thus L and W satisfy (63), and the proof is complete. □

6 Known Results

In this section, we collect some known facts about minimal realizations and positive real functions.

The following result is well known; for a proof, we refer the reader to [3, Section 2.7].

Theorem B *A real $m \times m$-matrix-valued rational function $H : \mathbb{C} \mapsto (\mathbb{C} \cup \infty)^{m \times m}$ is positive real if, and only if, the following conditions are satisfied :*

(i) *H has no poles in* $\mathbb{C}_+ := \{\, s \in \mathbb{C} \mid \operatorname{Re} s > 0 \,\}$*;*

(ii) $H(i\omega) + (H(i\omega))^H \geq 0$ *for all* $\omega \in \mathbb{R}$*, with* $i\omega$ *not a pole of* H*;*

(iii) *If* $i\omega_0$ *or* ∞ *is a pole of* H*, then it is a simple pole and the associated* $m \times m$ *residue matrix is Hermitian positive semi-definite.*

The next result is the well-known characterization of minimal realizations of descriptor systems; see, e.g., [17]. We also refer the reader to the related results on controllability, observability, and minimal realizations of descriptor systems given in [18, Chapter 2].

Theorem C *Let* H *be a real* $m \times m$*-matrix-valued rational function of the form*

$$H(s) = D + C^T\,(sE - A)^{-1}\,B, \tag{64}$$

where A *and* E *are* $n \times n$ *matrices. Then,* (64) *is a minimal realization of* H *if, and only if, the following five conditions are satisfied* :

(i) $\operatorname{rank}\,[\, A - sE \quad B \,] = n$ *for all* $s \in \mathbb{C}$*;*
(Finite controllability)

(ii) $\operatorname{rank}\,[\, E \quad B \,] = n$*;*
(Infinite controllability)

(iii) $\operatorname{rank}\,[\, A^T - sE^T \quad C \,] = n$ *for all* $s \in \mathbb{C}$*;*
(Finite observability)

(iv) $\operatorname{rank}\,[\, E^T \quad C \,] = n$*;*
(Infinite observability)

(v) $A \ker(E) \subseteq \operatorname{Im}(E)$.
(Absence of nondynamic modes)

If A and E have been brought to Weierstrass form, as described in Section 5.1, then Theorem C reduces as follows.

Theorem D *Let* H *be a real* $m \times m$*-matrix-valued rational function of the form*

$$H(s) = D + C_1^T\,(sI - A_1)^{-1}\,B_1 + C_2^T\,(sN - I)^{-1}\,B_2, \tag{65}$$

where A_1 *is an* $n_1 \times n_1$ *matrix and* N *is a nilpotent* $n_2 \times n_2$ *matrix consisting of only Jordan blocks. Then,* (65) *is a minimal realization of* H *if, and only if, the following five conditions are satisfied* :

(i) $\operatorname{rank}\,[\, A_1 - sI \quad B_1 \,] = n_1$ *for all* $s \in \mathbb{C}$*;*
(Finite controllability)

(ii) $\operatorname{rank}\,[\, N \quad B_2 \,] = n_2$*;*
(Infinite controllability)

(iii) $\operatorname{rank}\,[\, A_1^T - sI \quad C_1 \,] = n_1$ *for all* $s \in \mathbb{C}$*;*
(Finite observability)

(iv) $\operatorname{rank}\,[\,N^T \quad C_2\,] = n_2$;
(Infinite observability)

(v) *N has no 1×1 Jordan blocks.*
(Absence of nondynamic modes)

Note that conditions (i) and (ii) in Theorem D mean that

$$H_0(s) = C_1^T \left(sI - A_1\right)^{-1} B_1$$

is a minimal realization of H_0.

7 Concluding Remarks

We presented an extension of the classical positive real lemma for regular time-invariant linear systems to the descriptor case. We stress that the underlying LMIs only involve the given data matrices of the descriptor system. In particular, to formulate these LMIs, the matrix pencil of the descriptor system need not be in any canonical form, even though we employed a reduction to Weierstrass form in our proofs.

The results in this paper can be employed to check the passivity of reduced-order models of passive descriptor systems via the solution of LMIs. Furthermore, by allowing low-rank modifications of the matrix pencil of the reduced-order model, non-passive models can be turned into passive ones. Using the results in this paper, such low-rank modifications can be computed via bilinear semidefinite programs. This will be discussed in the forthcoming report [14].

References

[1] R. W. Freund, "Krylov-subspace methods for reduced-order modeling in circuit simulation," *J. Comput. Appl. Math.*, vol. 123, pp. 395–421, November 2000. Also available online from `http://cm.bell-labs.com/cs/doc/99`.

[2] Z. Bai and R. W. Freund, "A partial Padé-via-Lanczos method for reduced-order modeling," *Linear Algebra Appl.*, 2001, to appear.

[3] B. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis.* Englewood Cliffs, New Jersey: Prentice-Hall, 1973.

[4] Z. Bai and R. W. Freund, "Eigenvalue-based characterization and test for positive realness of scalar transfer functions," *IEEE Trans. Automat. Control*, vol. 45, December 2000, to appear.

[5] V. A. Jakubovič, "The solution of certain matrix inequalities in automatic control theory," *Soviet Math. Dokl.*, vol. 3, pp. 620–623, 1962.

[6] V. Popov, "Absolute stability of nonlinear systems of automatic control," *Automat. Remote Control*, vol. 22, pp. 857–875, 1962.

[7] R. E. Kalman, "Lyapunov functions for the problem of Lur'e in automatic control," *Proc. Nat. Acad. Sci. U.S.A.*, vol. 49, pp. 201–205, 1963.

[8] R. E. Kalman, “On a new characterization of linear passive systems,” in *Proceedings First Annual Allerton Conference on Circuit and System Theory*, pp. 456–470, University of Illinois, 1963.

[9] V. A. Jakubovič, “Solution of certain matrix inequalities encountered in nonlinear control theory,” *Soviet Math. Dokl.*, vol. 5, pp. 652–656, 1964.

[10] B. D. O. Anderson, “A system theory criterion for positive real matrices,” *SIAM J. Control.*, vol. 5, pp. 171–182, 1967.

[11] V. Popov, *Hyperstability of Control Systems.* Berlin, Germany: Springer-Verlag, 1973.

[12] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory.* Philadelphia, PA: SIAM Publications, 1994.

[13] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control.* Upper Saddle River, New Jersey: Prentice-Hall, 1996.

[14] R. W. Freund and F. Jarre, “Numerical computation of nearby positive real systems in the descriptor case,” numerical analysis manuscript, Bell Laboratories, Murray Hill, New Jersey, USA, 2001, in preparation.

[15] F. Gantmacher, *The Theory of Matrices*, vol. 2. New York: Chelsea, 1959.

[16] V. Ionescu, C. Oară, and M. Weiss, *Generalized Riccati Theory and Robust Control: A Popov Function Approach.* Chichester, England: John Wiley & Sons Ltd, 1999.

[17] G. C. Verghese, B. C. Lévy, and T. Kailath, “A generalized state-space for singular systems,” *IEEE Trans. Automat. Control*, vol. 26, pp. 811–831, August 1981.

[18] L. Dai, *Singular Control Systems*, vol. 118 of *Lecture Notes in Control and Information Sciences.* Berlin, Germany: Springer-Verlag, 1989.