A Sensitivity Result for Semidefinite Programs*

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Abstract

We study the sensitivity of unique and strictly complementary solutions of linear semidefinite programs when the data is subject to small arbitrary changes. We present an elementary and self-contained proof of the differentiability of the perturbations of the solutions as functions of the perturbations of the data, and we characterize the derivative as the solution of a system of linear equations. Such a differentiability result was obtained earlier by Shapiro in a more general setting and with a different characterization of the derivative, namely as the solution of a quadratic program. Our result improves upon earlier results that were established by Robinson for more general cone programs. We also present some examples that illustrate why our result does not apply to these more general cone programs.

Keywords: Linear semidefinite program, sensitivity, strict complementarity.

1 Introduction

We consider the perturbation of unique and strictly complementary optimal solutions of linear semidefinite programs when the data is subject to small arbitrary changes. Sensitivity results of this type have been derived for more general nonlinear programs with cone constraints. For such programs, Robinson [7, Theorem 2.3] established an upper semicontinuity property of the set of stationary points under certain regularity assumptions. In this paper, by exploiting the special structure of the semidefinite cone, we strengthen this result to a differentiability property, and we characterize the derivatives as solutions of a system of linear equations. We point out why this differentiability result for linear semidefinite programs does not hold for more general convex programs in conic form.

Our result complements earlier results in [3, 10, 11, 12] where perturbations of the objective function or of the right-hand side of a semidefinite program have been linked to optimal

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partitions. Further related results include the paper [4], where a local Lipschitz continuity property of unique and strictly complementary solutions of linear semidefinite programs is derived. The work of [5] analyzes the sensitivity of the optimal solutions of certain cone programs, which include semidefinite programs as a special case. In [1, 8, 9], the differentiability of certain solutions of nonlinear semidefinite programs is established under the assumption that a transversality condition and a second-order sufficient condition are satisfied. Furthermore, the derivative is characterized as the solution of a quadratic program. This result also applies to the class of linear semidefinite programs considered in Section 3 below.

In this paper, we present a brief and self-contained proof of differentiability of unique and strictly complementary solutions of linear semidefinite programs, and we derive a nonsingular system of linear equations for the derivatives. This characterization of the derivatives as solutions of a system of linear equations seems to be new. Computing the derivatives via simple linear system solves may be useful in the context of new approaches for solving more complicated nonlinear semidefinite programs by a sequence of linear semidefinite programs that are subject to small changes in the data from one iteration to the next. Such an approach will be described in the forthcoming report [2].

The remainder of this short note is as follows. In Section 2, we briefly introduce some notation and recall a known result for linear semidefinite programs. In Section 3 we present and prove our perturbation theorem. In Section 4 we discuss two examples.

2 Notation and a known result

The following notation has become standard in the literature on linear semidefinite programs. The space of real symmetric $n \times n$-matrices is denoted by $\mathcal{S}^n$. The notation $X \succeq 0$ ($X \succ 0$) is used to indicate that $X \in \mathcal{S}^n$ is positive semidefinite (positive definite). The standard scalar product on the space of $n \times n$-matrices is given by

$$\langle C, X \rangle := C \bullet X := \operatorname{trace}(C^T X) = \sum_{i,j=1}^{n} C_{i,j} X_{i,j}.$$

For given matrices $A^{(i)} \in \mathcal{S}^n$, $i = 1, 2, \ldots, m$, we define a linear map $\mathcal{A} : \mathcal{S}^n \to I\!R^m$ by

$$\mathcal{A}(X) := \begin{bmatrix} A^{(1)} \bullet X \\ \vdots \\ A^{(m)} \bullet X \end{bmatrix}, \quad X \in \mathcal{S}^n.$$

The adjoint operator $\mathcal{A}^* : I\!R^m \to \mathcal{S}^n$, which satisfies

$$\langle \mathcal{A}^*(y), X \rangle = y^T \mathcal{A}(X) \quad \text{for all} \quad X \in \mathcal{S}^n,\ y \in I\!R^m,$$

is given by

$$\mathcal{A}^*(y) = \sum_{i=1}^{m} y_i A^{(i)}, \quad y \in I\!R^m.$$

With these definitions, the standard pair of primal and dual linear semidefinite programs can now be stated as follows:

$$(P) \qquad \text{minimize} \quad C \bullet X \quad \text{subject to} \quad \mathcal{A}(X) = b, \quad X \succeq 0$$

and

$$(D) \qquad \text{maximize} \quad b^T y \quad \text{subject to} \quad \mathcal{A}^*(y) + S = C, \quad S \succeq 0.$$

Note that the data for (P) and (D) are the linear map $\mathcal{A}$, the vector $b \in \mathbb{R}^m$, and the matrix $C \in \mathcal{S}^n$. If there exists a matrix $X \succ 0$ that is feasible for (P), then we call X strictly feasible for (P) and say that (P) satisfies Slater's condition. Likewise, if there exists a matrix $S \succ 0$ and a vector y such that y, S is feasible for (D), we call (D) strictly feasible.

The following result on the pair of semidefinite programs (P) and (D) is well known; see, e.g., [6, 10]. If problem (P) or (D) satisfies Slater's condition, then the optimal values of (P) and (D) coincide. Furthermore, if in addition both problems (P) and (D) are feasible, then optimal solutions X^{opt} and y^{opt}, S^{opt} of both problems exist and satisfy the complementarity condition

$$X^{\text{opt}} S^{\text{opt}} = 0.$$

Conversely, if X and y, S are feasible points for (P) and (D), respectively, and if $XS = 0$, then X is an optimal solution of (P) and y, S is an optimal solution of (D).

3 A perturbation theorem

In this section, we present our result on the perturbation of strictly complementary solutions of pairs of linear semidefinite programs of the form (P) and (D).

Theorem 1 *Let a linear operator $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$, a vector $b \in \mathbb{R}^m$, and a matrix $C \in \mathcal{S}^n$ be the data of a pair (P) and (D) of primal and dual linear semidefinite programs. Assume that the programs (P) and (D) satisfy Slater's condition, and that $\bar{X} \in \mathcal{S}^n$ and $\bar{y} \in \mathbb{R}^m$, $\bar{S} \in \mathcal{S}^n$ are unique and strictly complementary solutions of (P) and (D), that is,*

$$\mathcal{A}(\bar{X}) = b, \quad \mathcal{A}^*(\bar{y}) + \bar{S} = C, \quad \bar{X}\bar{S} = 0, \quad \bar{X} \succeq 0, \quad \bar{S} \succeq 0, \quad \bar{X} + \bar{S} \succ 0. \tag{1}$$

If the data of (P) and (D) is changed by sufficiently small perturbations $\Delta\mathcal{A}$, Δb, and ΔC, then the optimal solutions of the perturbed semidefinite programs are differentiable functions of the perturbations. Furthermore, the derivatives

$$\dot{X} := D_{\mathcal{A},b,C}\bar{X}[\Delta\mathcal{A}, \Delta b, \Delta C], \quad \dot{y} := D_{\mathcal{A},b,C}\bar{y}[\Delta\mathcal{A}, \Delta b, \Delta C], \quad \textit{and} \quad \dot{S} := D_{\mathcal{A},b,C}\bar{S}[\Delta\mathcal{A}, \Delta b, \Delta C],$$

of the solution X, y, S at $\bar{X}$, $\bar{y}$, $\bar{S}$ satisfy

$$\begin{aligned} \mathcal{A}(\dot{X}) &= \Delta b - \Delta\mathcal{A}(\bar{X}), \\ \mathcal{A}^*(\dot{y}) + \dot{S} &= \Delta C - \Delta\mathcal{A}^*(\bar{y}), \\ \dot{X}\bar{S} + \bar{X}\dot{S} &= 0. \end{aligned} \tag{2}$$

Remark 1 In [1, Chapter 5.3] and [8, 9], more general sensitivity results for locally unique solutions of nonlinear, possibly nonconvex, semidefinite programs are derived under a transversality condition, a rank condition, strict complementarity, and a second-order sufficient condition. It is shown that derivatives can be characterized as optimal solutions of certain quadratic programs. These results also apply to the class of linear semidefinite programs considered here. The present paper, however, provides a simpler characterization of the derivatives as solutions of nonsingular systems of linear equations, and a self-contained direct proof.

Remark 2 A related sensitivity result for linear semidefinite programs for a more restricted class of perturbations than the ones considered in Theorem 1, but also under weaker assumptions, is given in [11].

Remark 3 We believe that the simple result of Theorem 1 has not been explicitly stated before in the literature. In particular, we are not aware of any earlier results that characterize the derivatives via the solution of a nonsingular system of linear equations, as stated in Corollary 1 below.

Remark 4 As indicated in Example 2 below, Theorem 1 contrasts recent sensitivity results in cone programming, see e.g. part (2) of Theorem 3.5.1 in [5].

Next, we present the proof of Theorem 1.

Proof. We first observe that the uniqueness of $\bar{X}$ and $\bar{y}$ implies that $\operatorname{rank}\mathcal{A} = m$. Indeed, assume that $\operatorname{rank}\mathcal{A}^* = \operatorname{rank}\mathcal{A} < m$. Then, there exists a vector $\Delta y \in I\!R^m$, $\Delta y \neq 0$, such that $\mathcal{A}^*(\Delta y) = 0$. If $b^T\Delta y = 0$, then $\bar{y} + \Delta y$ is also an optimal solution of (D) in contradiction to the uniqueness of $\bar{y}$. If $b^T\Delta y \neq 0$, then (D) does not have a finite optimal solution, which is again a contradiction. In particular, it follows that the dimension m of the variables $y \in I\!R^m$ of the dual problem (D) cannot be larger than the dimension $n(n+1)/2$ of the variables $X \in \mathcal{S}^n$ of the primal problem (P).

Slater's condition states that

$$\exists\, y,\ X \succ 0,\ S \succ 0: \quad \mathcal{A}(X) = b, \quad \mathcal{A}^*(y) + S = C.$$

By continuity and the observation that $\operatorname{rank}\mathcal{A} = m$, Slater's condition is also satisfied for all sufficiently small perturbations of the problem data. Hence, the perturbed problem possesses optimal solutions $\bar{X} + \Delta X$, $\bar{y} + \Delta y$, and $\bar{S} + \Delta S$. The optimality conditions of the perturbed problem are given by $\bar{X} + \Delta X \succeq 0$, $\bar{S} + \Delta S \succeq 0$, and

$$\begin{aligned} (\mathcal{A} + \Delta\mathcal{A})(\bar{X} + \Delta X) &= b + \Delta b, \\ (\mathcal{A}^* + \Delta\mathcal{A}^*)(\bar{y} + \Delta y) + S + \Delta S &= C + \Delta C, \\ (\bar{X} + \Delta X)(\bar{S} + \Delta S) &= 0. \end{aligned}$$

Subtracting from these equations the first three equations of (1) yields

$$\begin{aligned} (\mathcal{A} + \Delta\mathcal{A})(\Delta X) &= \Delta b - \Delta\mathcal{A}(\bar{X}), \\ (\mathcal{A}^* + \Delta\mathcal{A}^*)(\Delta y) + \Delta S &= \Delta C - \Delta\mathcal{A}^*(\bar{y}), \\ (\Delta X)\bar{S} + \bar{X}(\Delta S) &= -(\Delta X)(\Delta S). \end{aligned} \tag{3}$$

Note that the left-hand side of (3) represents an overdetermined system of linear equations for the unknowns ΔX, Δy, and ΔS. Since the existence of ΔX, Δy, and ΔS follows from Slater's condition, it suffices to verify uniqueness and differentiability. Neglecting the second-order terms in (3), we then obtain the result claimed in (2).

First, we show that (2) has a unique solution. By (1), $\bar{X}\bar{S} = 0 = \bar{S}\bar{X}$, and thus the matrices $\bar{X} \succeq 0$ and $\bar{S} \succeq 0$ commute. This guarantees that there exists a unitary matrix U and diagonal matrices

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \succeq 0 \quad \text{and} \quad \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \succeq 0 \tag{4}$$

such that

$$\bar{X} = U\Lambda U^T \quad \text{and} \quad \bar{S} = U\Sigma U^T. \tag{5}$$

By (1), we have $\bar{X} + \bar{S} \succ 0 = \bar{X}\bar{S}$. Together with (5), it follows that $\Lambda + \Sigma \succ 0 = \Lambda\Sigma$. This implies that there exists an integer $0 \le k \le n$ such that k of the diagonal entries of Λ are positive, $n-k$ of the diagonal entries of Σ are positive, and the two associated subsets of indices are disjoint. Thus, without loss of generality, we may assume that the diagonal entries in (4) satisfy

$$\lambda_1, \lambda_2, \ldots, \lambda_k > 0 \quad \text{and} \quad \sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_n > 0. \tag{6}$$

Next, we define matrices

$$\widetilde{\Delta S} := U^T \dot{S} U, \quad \widetilde{\Delta X} := U^T \dot{X} U,$$

and a linear map $\widetilde{\mathcal{A}} : \mathcal{S}^n \to I\!R^m$ by $\widetilde{\mathcal{A}}(Z) := \mathcal{A}(UZU^T)$ for all $Z \in \mathcal{S}^n$. It then follows that $\widetilde{\mathcal{A}}^*(y) = U^T \mathcal{A}^*(y) U$. With these definitions, the system (2) with all three right-hand sides set to zero is equivalent to the following system:

$$\begin{aligned} \widetilde{\mathcal{A}}(\widetilde{\Delta X}) &= 0, \\ \widetilde{\mathcal{A}}^*(\dot{y}) + \widetilde{\Delta S} &= 0, \\ \widetilde{\Delta X}\, \Sigma + \Lambda\, \widetilde{\Delta S} &= 0. \end{aligned} \tag{7}$$

For any $n \times n$-matrix Z, let $\Pi_{\text{up}}(Z)$ denote the upper triangular part of Z. By the symmetry of $\widetilde{\Delta X}$ and $\widetilde{\Delta S}$ and the fact that Λ and Σ are diagonal, it follows that the solution set of (7) does not change if the last set of equations is replaced by

$$\Pi_{\text{up}}\left(\widetilde{\Delta X}\, \Sigma + \Lambda\, \widetilde{\Delta S}\right) = 0. \tag{8}$$

Equation (8) and the first two equations of (7) form a linear system of $m + n(n+1)$ equations for $m + n(n+1)$ unknowns, namely the entries of the vector $\dot{y}$ and of the symmetric matrices $\widetilde{\Delta X}$ and $\widetilde{\Delta S}$. We now show that this linear system only has the zero solution. To this end, we only need to verify that $\widetilde{\Delta X} = 0$ and $\widetilde{\Delta S} = 0$. Indeed, since $\mathcal{A}^*$ has full column rank m, the second equation of (7) then implies that also $\dot{y} = 0$.

By the symmetry of $\widetilde{\Delta X}$ and the structure of Σ, the trailing $(n-k) \times (n-k)$-block of $\widetilde{\Delta X}$ is zero. Likewise, the leading $k \times k$-block of $\widetilde{\Delta S}$ is zero. It also turns out $\widetilde{\Delta X}_{i,j} = 0$ for all $1 \le i \le k < j \le n$. Indeed, suppose there were indices $1 \le i \le k < j \le n$ such that $\widetilde{\Delta X}_{i,j} \ne 0$. By (8), we have

$$\sigma_j\, \widetilde{\Delta X}_{i,j} + \lambda_i\, \widetilde{\Delta S}_{i,j} = 0,$$

and together with (6), it would follow that for all such indices $1 \le i \le k < j \le n$, the matrix entries $\widetilde{\Delta S}_{i,j}$ and $\widetilde{\Delta X}_{i,j}$ had opposite signs. This, in turn, would imply

$$0 > 2 \sum_{i \le k < j} \widetilde{\Delta X}_{i,j}\, \widetilde{\Delta S}_{i,j} = \langle \widetilde{\Delta X}, \widetilde{\Delta S} \rangle = -\langle \widetilde{\Delta X}, \widetilde{\mathcal{A}}^*(\dot{y}) \rangle = -\langle \widetilde{\mathcal{A}}(\widetilde{\Delta X}), \dot{y} \rangle = \langle 0, \dot{y} \rangle = 0,$$

which is a contradiction. Finally, if for some $1 \le i, j \le k$ there was an entry $\widetilde{\Delta X}_{i,j} \ne 0$, then $\bar{X} + \epsilon U \widetilde{\Delta X} U^T$ and $\bar{y}$, $\bar{S}$ would be optimal solutions of the pair of programs (P) and (D) for sufficiently small $|\epsilon|$, since the entries $\widetilde{\Delta X}_{i,j}$ only change the strictly positive eigenvalues of $\bar{X}$ for $1 \leq i, j \leq k$. This contradicts the assumption of the uniqueness of $\bar{X}$. Likewise, it follows that $\widetilde{\Delta S}_{i,j} = 0$ for $k < i, j \leq n$.

We have thus shown that the overdetermined system (2) has a unique solution. In fact, we have shown that the system that is obtained when the last row of (2) is replaced by

$$\Pi_{\text{up}} \left(U^T \dot{X} \bar{S} U + U^T \bar{X} \dot{S} U \right) = 0.$$

also has a unique solution. With U denoting the fixed matrix from (5), we can now apply the implicit function theorem to the system

$$\mathcal{A}(X) - b = 0, \quad \mathcal{A}^*(y) - C + S = 0, \quad \Pi_{\text{up}} \left(U^T X S U \right) = 0. \tag{9}$$

As we have just seen, the linearization of (9) at $\bar{X}$, $\bar{y}$, $\bar{S}$ is nonsingular, and hence (9) has a differentiable and locally unique solution. Since the solutions of (2) also satisfy (9), the claim of the theorem follows. $\square$

In Theorem 1, the derivatives $\dot{X}$, $\dot{y}$, and $\dot{S}$ are characterized via an overdetermined system of linear equations. The preceding proof also shows that these derivatives are given as a solution of a nonsingular system of linear equations. For completeness, this system is stated explicitly in the following corollary.

Corollary 1 *Let the assumptions of Theorem 1 be satisfied, and let U be an orthogonal matrix that simultaneously diagonalizes $\bar{X}$ and $\bar{S}$, as in (5). Then, the derivatives $\dot{X}$, $\dot{y}$, and $\dot{S}$ are given as the unique solution of the following nonsingular system of $m+n(n+1)$ linear equations for $m + n(n + 1)$ unknowns:*

$$\begin{aligned} \mathcal{A}(\dot{X}) &= \Delta b - \Delta\mathcal{A}(\bar{X}), \\ \mathcal{A}^*(\dot{y}) + \dot{S} &= \Delta C - \Delta\mathcal{A}^*(\bar{y}), \\ \Pi_{\text{up}}(U^T(\dot{X}\bar{S} + \bar{X}\dot{S})U) &= 0. \end{aligned} \tag{10}$$

Here, $\Pi_{\text{up}}(U^T(\dot{X}\bar{S} + \bar{X}\dot{S})U)$ denotes the upper triangular part of the matrix $U^T(\dot{X}\bar{S} + \bar{X}\dot{S})U$.

4 Discussion

The above proof relies on the fact that the optimal solution is assumed to be strictly complementary. The concept of strict complementarity is based on the identification of smooth active constraint manifolds, such as the zero eigenvalues for linear semidefinite programs. For general closed convex cones such constraint manifolds are not as easily defined as for the positive orthant or the semidefinite cone. Next, we present an example that illustrates this difficulty.

Example 1 We consider the set $\mathcal{C} \subset I\!R^2$ defined by the intersection of the half-planes containing the origin and that are tangent to the unit circle at the coordinates

$$x_2 = \pm\frac{1}{2^n} \quad \text{and} \quad x_1 = \pm\sqrt{1 - x_2^2} \quad \text{for} \quad n \in I\!N. \tag{11}$$

Note that $\mathcal{C}$ is a closed and bounded convex set. It is constructed from a square with the vertices $[0, \pm\sqrt{2}]^T$ and $[\pm\sqrt{2}, 0]^T$. The vertices on the x_1-axis are cut off by an infinite number of 'facet-defining' cuts. None of these facets contains the point $\bar{x} := [1, 0]^T$ of $\mathcal{C}$. Nevertheless, $\bar{x}$ is the unique optimal solution of the convex program

$$\text{minimize } [-1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{subject to} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{C}. \tag{12}$$

By adding the redundant constraint

$$[1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1 \quad \text{with multiplier} \quad y_0 := 1,$$

and defining the multipliers $y_i = 0$ for all constraints in (11), the point $\bar{x}$ can be complemented to a 'formally strictly complementary' solution. The reason is that none of the constraints in (11) is active at the point $\bar{x}$, that is, satisfied with equality. However, the differentiability result of Theorem 1 does not hold in this situation. There are arbitrarily small perturbations $\Delta c = [0, \epsilon]^T$ of the objective vector $c := [-1, 0]^T$ for which the set of optimal solutions contains more than one point. Furthermore, for those arbitrarily small perturbations that do have unique solutions $\bar{x} + \Delta x$, the solutions jump in a nondifferentiable fashion from one vertex of $\mathcal{C}$ to the next. Indeed, each time the solution jumps, the norm $\|\Delta x\|_2$ changes by a factor of about 3/2, and hence $\|\Delta x\|_2$ is not a differentiable function of Δc.

Similarly, the closed convex cone

$$\mathcal{K} := \{\, x \in I\!R^3 \mid x_3 > 0,\ (x_1, x_2)/x_3 \in \mathcal{C} \,\} \cup \{0\}$$

does not possess a straightforward generalization of active constraint manifolds. Problem (12) can be rewritten as

$$\text{minimize} \quad x_1 \quad \text{subject to} \quad x_3 = 1, \quad x \in \mathcal{K},$$

which is of the form (P) with the positive semidefiniteness constraint $X \succeq 0$ replaced by the cone constraint $x \in \mathcal{K}$. This example shows that the differentiability properties stated in Theorem 1 cannot be generalized in a straightforward fashion to more general convex cone programs.

It is well known that unique—and therefore strictly complementary—solutions of a primal-dual pair of linear programs do not change under sufficiently small perturbations of the problem data. For linear semidefinite programs, however, the situation is more complicated. In the appendix of [11], it is shown that the optimal partition of a strictly complementary solution of a primal-dual pair of linear semidefinite programs (P) and (D) can indeed change under arbitrarily small perturbations of the problem data C or b, even if Slater's condition is satisfied for (P) and (D). However, if, in addition, the optimal solution is unique, Theorem 1 of the present paper is applicable. This situation is illustrated in the next example.

Example 2 In a small neighborhood of the point $\bar{x} = [1, 0]^T$, the set $\mathcal{C}$ of Example 1 forms a very close approximation to the unit circle. To contrast the behavior of Example 1 with our sensitivity result, we replace the set $\mathcal{C}$ by the unit circle itself. We then obtain the semidefinite program of form (P),

$$\text{minimize} \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \bullet X \quad \text{subject to} \quad X = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_1 & x_2 & 1 \end{bmatrix}, \quad X \succeq 0.$$

The feasible solutions are all x_1, x_2 in the unit ball of $I\!R^2$, and the unique optimal solution $\bar{X}$ is given by $\bar{x}_1 = 1$, $\bar{x}_2 = 0$, just as in (12). Moreover, the objective function coincides with the one from (12) up to a factor of 2, which is introduced merely for notational convenience. The dual problem is as follows:

$$\text{maximize} \quad y_1 + y_2 + y_3 \quad \text{subject to} \quad \begin{bmatrix} y_1 & y_4 & 0 \\ y_4 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Its unique optimal solution is given by $\bar{y}_1 = \bar{y}_3 = -1$ and $\bar{y}_2 = \bar{y}_4 = 0$. The corresponding optimal dual slack is

$$\bar{S} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

and is strictly complementary, that is,

$$\bar{X} + \bar{S} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \succ 0.$$

Thus, Theorem 1 is applicable. For the perturbation

$$\Delta C := - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{bmatrix},$$

we set

$$\dot{X} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{bmatrix}, \quad \dot{y} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\epsilon \end{bmatrix}, \quad \text{and} \quad \dot{S} := \begin{bmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & -\epsilon \\ 0 & -\epsilon & 0 \end{bmatrix}.$$

Then, $\dot{X}$, $\dot{y}$, and $\dot{S}$ satisfy all conditions in (2). In particular, in contrast to the case of linear programming, the optimal solution X will change for any $\epsilon \neq 0$.

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