AN INTERIOR-POINT METHOD FOR CONVEX FRACTIONAL PROGRAMMING

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Abstract

We present an interior-point method for convex fractional programming. The proposed algorithm converges in polynomial time, just as in the case of a convex problem, even though convex fractional programs are only pseudo-convex. More precisely, the rate of convergence is measured in terms of the area of two-dimensional convex sets $\mathcal{C}_k$ containing the optimal points, and the area of $\mathcal{C}_k$ is reduced by a constant factor $c < 1$ at each iteration. The factor c depends only on the self-concordance parameter of a barrier function associated with the feasible set. We present an outline of a practical implementation of the proposed method, and we report results of a few numerical experiments.

1. Introduction

Interior-point methods for the solution of nonlinear programming problems were already introduced in the 1950s and 1960s; see [6] and the references given there. In the 1970s, new and seemingly superior approaches, such as sequential quadratic programming techniques, were developed, and as a result, interior-point methods were largely discarded. In 1984, Karmarkar [14] announced a practical polynomial-time interior-point algorithm for linear programs. His landmark paper triggered a true revival of interior-point techniques. Since 1984, numerous interior-point methods for linear and nonlinear programming problems have been proposed and studied extensively; see, e.g., [24] and the references given there. In contrast to linear programs, interior-point methods for nonlinear programs converge in polynomial time only in special cases. For example, polynomial-time methods exist for classes of convex programming problems [22, 17, 10, 11, 4] and for optimization problems over cones of positive semidefinite matrices [1, 12]. Recently, Nesterov and Nemirovsky [18] developed an elegant general framework for polynomial-time interior point methods, based on the concept of self-concordant barrier functions.

1.1. The problem

In this paper, we extend the approach of [18], and we present a polynomial-time interior-point method for *convex fractional programs* of the type

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \frac{x_2}{x_1} \qquad \text{subject to} \quad f_i(x) \leq 0, \quad i = 1, 2, \ldots, s, \tag{1.1}$$

or

$$\underset{x \in \mathbb{R}^n}{\text{maximize}} \ \frac{x_2}{x_1} \qquad \text{subject to} \quad f_i(x) \leq 0, \quad i = 1, 2, \ldots, s, \tag{1.2}$$

with constraint functions $f_i : \mathbb{R}^n \mapsto \mathbb{R} \cup \infty$. We denote by

$$\mathcal{S} := \{ \, x \in \mathbb{R}^n \mid f_i(x) \leq 0, \ i = 1, 2, \ldots, s \, \} \tag{1.3}$$

the feasible domain of (1.1) and (1.2). We always assume that $\mathcal{S}$ is convex with a nonempty interior, and that

$$(x_1, x_2) \neq (0, 0) \quad \text{for all} \quad x \in \mathcal{S}. \tag{1.4}$$

We stress that the objective function $f_0(x) = x_2/x_1$ is not convex, and hence neither (1.1) nor (1.2) are convex programming problems. However, as we will discuss in Section 5.1, the function f_0 is pseudo-convex if the feasible set $\mathcal{S}$ satisfies the additional condition

$$x_1, \ x_2 \ \geq \ 0 \quad \text{for all} \quad x \in \mathcal{S}. \tag{1.5}$$

In Section 2.2, we will show that any convex fractional program (1.1) can always be reduced to a problem of the same form (1.1) such that $\mathcal{S}$ fulfills (1.5). Furthermore, we will see that problem (1.2) can be rewritten in the form (1.1).

1.2. Applications

Problems of the form (1.1) and (1.2) include a number of important special cases.

First, consider optimization problems of the type

$$\underset{x \in \mathbb{R}^n}{\text{maximize}} \ \frac{g(x)}{h(x)} \qquad \text{subject to} \quad x \in \mathcal{S}, \tag{1.6}$$

where the denominator function h satisfies $h(x) > 0$ for all feasible points $x \in \mathcal{S}$. The problem (1.6) is called a *concave-convex fractional program* [19] if the numerator function g is concave, h is convex, and the constraint functions f_i, $i = 1, 2, \ldots, s$, in (1.3) are all convex. Concave-convex fractional programs arise in many applications; see [19, 20, 21] and the references given there. Clearly, the concave-convex fractional program (1.6) is equivalent to the problem

$$\begin{aligned} &\underset{\alpha,\, \beta \in \mathbb{R},\, x \in \mathbb{R}^n}{\text{maximize}} \ \frac{\beta}{\alpha} \\ &\text{subject to} \quad h(x) - \alpha \leq 0, \ \beta - g(x) \leq 0, \ f_i(x) \leq 0, \quad i = 1, 2, \ldots, s, \end{aligned}$$

which is a convex fractional program of the type (1.2).

Second, consider the problem of minimizing the Euclidean condition number of positive definite matrices $M(x)$ that depend affinely on $x \in I\!R^n$. Let $M_0, M_1, \ldots, M_n$ be given real symmetric $l \times l$ matrices, and set

$$M(x) := M_0 + \sum_{j=1}^{n} x_j M_j. \tag{1.7}$$

Moreover, we require that

$$M(x) \neq 0 \quad \text{for all} \quad x \in I\!R^n \tag{1.8}$$

$$\text{and} \quad M(x^{(0)}) > 0 \quad \text{for at least one} \quad x^{(0)} \in I\!R^n. \tag{1.9}$$

Here and in the sequel, the notation $M > 0$ ($M \geq 0$) is used to indicate that M is positive definite (positive semidefinite). The problem then is to find, among all positive semidefinite matrices $M(x)$, one with minimal condition number:

$$\underset{x \in I\!R^n}{\text{minimize}} \quad \frac{\lambda_{\max}(M(x))}{\lambda_{\min}(M(x))} \quad \text{subject to} \quad M(x) \geq 0. \tag{1.10}$$

Here, $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the largest and smallest eigenvalue of M. Applications that lead to problems of the type (1.10) are described in [5, 9]. Setting, for symmetric matrices M,

$$\det{}_+(M) := \begin{cases} \det(M) & \text{if } M \geq 0, \\ -\infty & \text{otherwise,} \end{cases} \tag{1.11}$$

we can rewrite (1.10) as the following optimization problem:

$$\begin{aligned} &\underset{\alpha,\, \beta \in I\!R,\, x \in I\!R^n}{\text{minimize}} \quad \frac{\beta}{\alpha} \\ &\text{subject to} \quad -\alpha \leq 0, \; -\det{}_+(M(x) - \alpha I) \leq 0, \; -\det{}_+(\beta I - M(x)) \leq 0. \end{aligned} \tag{1.12}$$

Here, I denotes the $l \times l$ identity matrix. Clearly, the problem (1.12) is a convex fractional program of the type (1.1). The condition (1.9) guarantees that the feasible set of (1.12) has a nonempty interior. Note that, in (1.12), $\alpha = \beta = 0$ is feasible if, and only if, there exists a point $x \in I\!R^n$ with $M(x) = 0$. Therefore, the assumption (1.8) guarantees that the condition (1.4) is satisfied for (1.12). If $M(x)$ is a linear combination of matrices, i.e., $M_0 = 0$ in (1.7), then we can fix $\alpha = 1$ in (1.12). Thus, for this special case, the problem (1.12) is actually a convex program, and it can be solved by the interior-point methods described in [18, 12]. We remark that the interior-point method recently proposed by Boyd and El Ghaoui [3] can also be applied to problems of the type (1.10). However, the method in [3] does not converge in polynomial time.

1.3. Outline

The remainder of this paper is organized as follows. In Section 2, we first recall some known results about self-concordant functions, and then we summarize our assumptions on (1.1) and (1.2). In Section 3, we present our interior-point method for convex fractional programs, and we derive some of its basic properties. In Section 4, we show that this algorithm converges in polynomial time. In Section 5, we first recall the optimality conditions for convex fractional programs, and we then derive a stopping criterion for the proposed method. In Section 6, we sketch a practical implementation. In Section 7, results of a few numerical experiments are reported. Finally, in Section 8, we make some concluding remarks.

1.4. Notations

Throughout this paper, all vectors and matrices are assumed to be real. We will use the following notations. We denote by $M^T = [\, m_{kj} \,]$ the transpose of the matrix $M = [\, m_{jk} \,]$. The vector norm $\|x\| := \sqrt{x^T x}$ is always the Euclidean norm, and $\|M\| := \max_{\|x\|=1} \|Mx\|$ is the associated matrix norm. For symmetric positive definite matrices M, we denote by $\|x\|_M := \sqrt{x^T M x}$ the vector norm induced by M. For symmetric matrices M and N, the notation $M > N$, respectively $M \geq N$, is used to indicate that $M - N$ is positive definite, respectively positive semidefinite. For vectors $x \in \mathbb{R}^n$, $n \geq 2$, we always denote by x_1 and x_2 the first two components of x, and

$$x_{1,2} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

is the truncated vector with components x_1 and x_2. Finally, for sets $\mathcal{C} \in \mathbb{R}^m$, we denote by $\mathcal{C}^\circ$ the interior (with respect to the Euclidean topology of $\mathbb{R}^m$) of $\mathcal{C}$.

2. Self-concordant functions and basic assumptions

Self-concordant functions are defined and examined in great detail by Nesterov and Nemirovsky in [17, 18]. In this section, we first collect some known results about self-concordant functions, which will be needed later on, and then we state our basic assumptions on (1.1) and (1.2).

2.1. Results about self-concordant functions

Let $\mathcal{S} \subset \mathbb{R}^n$ be a given convex set with nonempty interior $\mathcal{S}^\circ$. Let $\phi : \mathcal{S}^\circ \mapsto \mathbb{R}$ be a given function that is assumed to be three times continuously differentiable on $\mathcal{S}^\circ$. For each pair of points $x \in \mathcal{S}^\circ$ and $h \in \mathbb{R}^n$, we define a function $\psi : \mathbb{R} \mapsto \mathbb{R}$ by setting

$$\psi(\beta) = \psi_{x,h}(\beta) := \phi(x + \beta h) \quad \text{for all} \quad \beta \in \mathbb{R}. \tag{2.1}$$

2.1.1. Self-concordance

In slight variation to the definition in [18], we call the function ϕ *self-concordant* if ϕ is convex and the functions $\psi = \psi_{x,h}$ satisfy

$$|\psi'''(0)| \;\leq\; 2\psi''(0)^{3/2} \quad \text{for all} \quad x \in \mathcal{S}^\circ \quad \text{and} \quad h \in \mathbb{R}^n. \tag{2.2}$$

A self-concordant function ϕ is said to be *strongly self-concordant* if $\lim_{i\to\infty} \phi(x^{(i)}) = \infty$ for every sequence $\{x^{(i)}\}_{i=o}^{\infty} \subset \mathcal{S}^\circ$ that converges to a boundary point of $\mathcal{S}$. Note that the condition (2.2) just states that the *relative* change of $\nabla^2\phi(x)$ is bounded in a small neighborhood of x (since ψ''', which describes the absolute change of ψ'', is bounded by a suitable power of ψ'').

In the following, we always assume that the function ϕ is strongly self-concordant. Next, we list a number of properties that follow from the strong self-concordance of ϕ; we refer the reader to [18, 11] for proofs of these results.

2.1.2. Inner ellipsoid

First, we note that ϕ is strictly convex on $\mathcal{S}^\circ$. Therefore, for all $x \in \mathcal{S}^\circ$, the Hessian $H_x := \nabla^2\phi(x)$ of ϕ is positive definite, and we can associate the canonical norm $\|\cdot\|_{H_x}$ with x. Furthermore, for each point $x \in \mathcal{S}^\circ$, the unit ball with respect to the H_x-norm describes an *inner ellipsoid* [22, 23] for $\mathcal{S}$. More precisely, we have

$$x + h \in \mathcal{S} \quad \text{for all} \quad x \in \mathcal{S}^\circ \quad \text{and all} \quad h \in I\!R^n \quad \text{with} \quad \|h\|_{H_x} \leq 1. \tag{2.3}$$

2.1.3. Newton's method

Self-concordance of ϕ implies that Newton's method without line search converges "well" (in a sense to be specified below) when applied to the unconstrained minimization problem

$$\operatorname*{minimize}_{x \in \mathcal{S}} \ \phi(x). \tag{2.4}$$

We now assume that $\mathcal{S}$ is a bounded set in $I\!R^n$. Since ϕ is strictly convex and strongly self-concordant, it follows that the problem (2.4) has a unique optimal solution $x_\star$ and $x_\star \in \mathcal{S}^\circ$; the point $x_\star$ is called the *analytic center* [22] of $\mathcal{S}$. Let $x \in \mathcal{S}^\circ$, and let

$$\Delta x := -H_x^{-1}\nabla\phi(x) \tag{2.5}$$

denote the Newton step for obtaining the next Newton iterate $\tilde{x} := x + \Delta x$. If $\|\Delta x\|_{H_x} \leq 1/4$, then

$$\tilde{x} \in \mathcal{S}^\circ \tag{2.6}$$

and Newton's method is guaranteed to be quadratically convergent. More precisely, with $\Delta\tilde{x}$ denoting the next Newton step starting at $\tilde{x}$, we have

$$\|\Delta\tilde{x}\|_{H_{\tilde{x}}} \ \leq \ \frac{\|\Delta x\|_{H_x}^2}{(1 - \|\Delta x\|_{H_x})^2}. \tag{2.7}$$

Furthermore, if $\|\Delta x\|_{H_x} < 1/3$, then the distance of x to $x_\star$ is bounded as follows:

$$\|x - x_\star\|_{H_x} \ \leq \ 1 - (1 - 3\|\Delta x\|_{H_x})^{1/3}. \tag{2.8}$$

The importance of these results on Newton's method is that they do not depend at all on the data of the problem.

We remark that, by (2.5), the H_x-norm $\|\Delta x\|_{H_x}$ of the Newton step is the same as the H_x^{-1}-norm of the gradient:

$$\|\Delta x\|_{H_x} = \|\nabla\phi(x)\|_{H_x^{-1}}. \tag{2.9}$$

The equivalence (2.9) will frequently be used later on.

2.1.4. m-self-concordance

In the following, we assume that the functions $\psi = \psi_{x,h}$ defined in (2.1), in addition to (2.2), satisfy a second condition, which was also introduced by Nesterov and Nemirovsky [18]. We call the self-concordant function ϕ *m-self-concordant* if there is a constant $m > 0$ such that

$$|\psi'(0)| \ \leq \ \sqrt{m}\sqrt{\psi''(0)} \quad \text{for all} \quad x \in \mathcal{S}^\circ \quad \text{and} \quad h \in I\!R^n. \tag{2.10}$$

An equivalent formulation of the condition (2.10) is that the Newton step (2.5) satisfies

$$\|\Delta x\|_{H_x} \;\leq\; \sqrt{m} \quad \text{for all} \quad x \in \mathcal{S}^\circ. \tag{2.11}$$

Note that the inequalities (2.2) and (2.10) for the functions $\psi = \psi_{x,h}$ are assumed to hold only at $\beta = 0$. However, since they are required to be satisfied for all $x \in \mathcal{S}^\circ$, it readily follows that the conditions (2.2) and (2.10) are actually fulfilled for all β with $x + \beta h \in \mathcal{S}^\circ$. For example, we have that, for all $x \in \mathcal{S}^\circ$ and $h \in \mathbb{R}^n$,

$$|\psi'_{x,h}(\beta)| \;\leq\; \sqrt{m}\sqrt{\psi''_{x,h}(\beta)} \quad \text{for all} \quad \beta \in \mathbb{R} \quad \text{with} \quad x + \beta h \in \mathcal{S}^\circ. \tag{2.12}$$

In Section 4, we will need this more general formulation of (2.10).

2.1.5. Outer ellipsoid

For bounded sets $\mathcal{S}$, the m-self-concordance of the function ϕ implies that we have the following *outer ellipsoid* for $\mathcal{S}$ centered at the minimizer $x_\star$ of (2.4):

$$\mathcal{S} \;\subseteq\; x_\star + \left\{ h \in \mathbb{R}^n \;\middle|\; \|h\|_{H_{x_\star}} \leq m + 2\sqrt{m} \right\}. \tag{2.13}$$

For a proof of (2.13), see [13].

2.1.6. Sums of self-concordant functions

Finally, we will make use of the following result.

Lemma 2.1. *If ϕ_j, $j = 1, 2$, are m_j-self-concordant functions on $\mathcal{S}^\circ$, then the function $\phi := \phi_1 + \phi_2$ is $(m_1 + m_2)$-self-concordant on $\mathcal{S}^\circ$.*

Proof. In [18], it is shown that the sum of self-concordant functions is again self-concordant. Thus it only remains to verify that the functions $\psi = \psi_{x,h} := \psi_1 + \psi_2$ associated with $\phi = \phi_1 + \phi_2$ satisfies (2.10) with $m := m_1 + m_2$. Using the inequality

$$(a + b)^2 \;\leq\; (m_1 + m_2)\left(\frac{a^2}{m_1} + \frac{b^2}{m_2}\right) \quad \text{for all} \quad a, b \in \mathbb{R},$$

and the conditions (2.10) for the functions ψ_j associated with ϕ_j, $j = 1, 2$, we obtain

$$\begin{aligned} |\psi'(0)|^2 \;\leq\; \left(|\psi'_1(0)| + |\psi'_2(0)|\right)^2 \;&\leq\; (m_1 + m_2)\left(\frac{|\psi'_1(0)|^2}{m_1} + \frac{|\psi'_2(0)|^2}{m_2}\right) \\ &\leq\; (m_1 + m_2)\left(\psi''_1(0) + \psi''_2(0)\right) = m\,\psi''(0). \end{aligned}$$

Taking square roots in this inequality gives (2.10). □

2.2. Basic assumptions

We now state our assumptions on the convex fractional program (1.1). Note that, since $\max(x_2/x_1) = -\min(-x_2/x_1)$, the problem (1.2) can always be written as a convex fractional program (1.1) by setting $x_2 := -x_2$. Therefore, from now on, we only consider problems of the form (1.1).

Throughout this paper, we assume that the feasible domain $\mathcal{S}$ of (1.1) is a bounded convex set in $I\!R^n$. Furthermore, we always require that $\mathcal{S}$ has a nonempty interior, i.e.,

$$\mathcal{S}^\circ \neq \emptyset, \tag{2.14}$$

and that $\mathcal{S}$ satisfies the condition (1.4). The constraint functions f_i, $i = 1, 2, \ldots, s$, are assumed to be three times continuously differentiable on $\mathcal{S}^\circ$.

2.2.1. Logarithmic barrier function

We associate with $\mathcal{S}$ the *logarithmic barrier function* ϕ defined by

$$\phi(x) := -\sum_{i=1}^{s} \log(-f_i(x)) \quad \text{for all} \quad x \in \mathcal{S}^\circ. \tag{2.15}$$

We assume that the function ϕ is m-self-concordant for some constant number $m \geq 1$. We refer to m as the *self-concordance parameter* of ϕ. If all constraint functions f_i are convex, then one can choose $m = s$ as the self-concordance parameter; see, e.g., [18, 11]. Recall that s is the number of constraints in (1.1). We stress that ϕ is indeed m-self-concordant in many important cases. For example, this condition is satisfied if the functions f_i are all linear or convex quadratic; see [18, 11]. Furthermore, in [18], it is shown that the logarithmic barrier function $-\log(\det_+(M))$ corresponding to a single constraint function of the type (1.11) is l-self-concordant on the cone of positive definite $l \times l$ matrices M. We remark that the function $-(\det_+(M))^{1/l}$, but not $-\det_+(M)$ itself, is convex on the cone of positive definite $l \times l$ matrices M, and thus each constraint function $-\det_+(M)$ counts as l convex constraints; see also [12]. Consequently, the logarithmic barrier function ϕ associated with the convex fractional program (1.12) is m-self-concordant with self-concordance parameter $m = 2l + 1$.

Finally, we remark that any m-self-concordant logarithmic barrier function (2.15) is always strongly m-self-concordant since, by construction, $\phi(x)$ tends to infinity as x approaches the boundary of $\mathcal{S}$.

2.2.2. Reduction to the positive quadrant

The four quadrants of $I\!R^2$ induce the following decomposition of the feasible domain of (1.1):

$$\mathcal{S} = \mathcal{S}_I \cup \mathcal{S}_{II} \cup \mathcal{S}_{III} \cup \mathcal{S}_{IV}, \tag{2.16}$$

where

$$\begin{aligned} \mathcal{S}_I &:= \{\, x \in \mathcal{S} \mid x_1 \geq 0,\ x_2 \geq 0 \,\}, \quad \mathcal{S}_{II} := \{\, x \in \mathcal{S} \mid x_1 \leq 0,\ x_2 \geq 0 \,\}, \\ \mathcal{S}_{III} &:= \{\, x \in \mathcal{S} \mid x_1 \leq 0,\ x_2 \leq 0 \,\}, \quad \mathcal{S}_{IV} := \{\, x \in \mathcal{S} \mid x_1 \geq 0,\ x_2 \leq 0 \,\}. \end{aligned} \tag{2.17}$$

From (1.4) and the convexity of $\mathcal{S}$, it follows that at most three of the sets (2.17) are nonempty. Determining whether the set $\mathcal{S}_J$, $J \in \{\, I, II, III, IV \,\}$, is nonempty is a convex programming problem that can be solved in polynomial time by using any of the algorithms described in [18, 11]. Therefore, by solving at most four convex programming problems, we can find out which of the four components of $\mathcal{S}$ in (2.16) are nonempty.

Once the nonempty components of $\mathcal{S}$ are known, the search for an optimal solution of (1.1) can be restricted to precisely one of the four sets (2.17). This is trivially true if only one set is nonempty. For the case that two or three sets are nonempty, note that the objective function $f_0(x) = x_2/x_1$ of (1.1) is nonnegative on $\mathcal{S}_I$ and $\mathcal{S}_{III}$, and f_0 is nonpositive on $\mathcal{S}_{II}$ and $\mathcal{S}_{IV}$. Hence, if $\mathcal{S}$ is contained in either $\mathcal{S}_I \cup \mathcal{S}_{IV}$ or in $\mathcal{S}_{II} \cup \mathcal{S}_{III}$, then we only need to consider $\mathcal{S}_{IV}$ or $\mathcal{S}_{II}$, respectively. In all other cases, it follows that $\mathcal{S}^\circ \cap \{ x \in I\!\!R^n \mid x_1 = 0 \} \neq \emptyset$, and as a result, the infimum of the objective function f_0 on $\mathcal{S}$ is $-\infty$. In this case, we can stop.

These considerations show that, without loss of generality, we can always assume that the feasible domain $\mathcal{S}$ of (1.1) is one of the four sets (2.17). We can even set

$$\mathcal{S} := \mathcal{S}_I. \tag{2.18}$$

Indeed, since $\min(x_2/x_1) = -1/\min(-x_1/x_2)$, the case $\mathcal{S} = \mathcal{S}_{II}$ can always be transformed into (2.18) by redefining the variables x_1 and x_2 as follows: $x_2 := -x_1$, $x_1 := x_2$. Similarly, the case $\mathcal{S} = \mathcal{S}_{III}$ and $\mathcal{S} = \mathcal{S}_{IV}$ can be written in the form (2.18) by setting $x_1 := -x_1$, $x_2 := -x_2$ and $x_2 := x_1$, $x_1 := -x_2$, respectively. Therefore, from now on, we assume that $\mathcal{S} = \mathcal{S}_I$. This, together with (1.4), guarantees that the feasible domain $\mathcal{S}$ always satisfies the following conditions:

$$x_1,\, x_2 \;\geq\; 0 \quad \text{and} \quad (x_1, x_2) \neq (0,0) \quad \text{for all} \quad x \in \mathcal{S}. \tag{2.19}$$

3. A method of centers

In this section, we describe our interior-point method for convex fractional programs, and we derive some of its basic properties. We stress that this algorithm is merely of theoretical interest since it allows a simple proof of polynomial convergence. A practical implementation of the method will be sketched in Section 6.

3.1. An equivalent formulation

Our interior-point method is based on the following reformulation of the convex fractional program (1.1). We set

$$v(t) := \begin{bmatrix} -\sin t \\ \cos t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in I\!\!R^n \quad \text{and} \quad u(x,t) := x^T v(t), \quad x \in I\!\!R^n, \quad t \in [0, \pi/2]. \tag{3.1}$$

In view of (2.19), we have, for all $x \in \mathcal{S}$ and $t \in [0, \pi/2]$,

$$u(x,t) \;\leq\; 0 \quad \Longleftrightarrow \quad \frac{x_2}{x_1} \;\leq\; \tan t, \tag{3.2}$$

and thus (1.1) is equivalent to the following problem:

$$\underset{t \in [0,\pi/2],\, x \in I\!\!R^n}{\text{minimize}} \;\; t \qquad \text{subject to} \quad u(x,t) \;\leq\; 0, \quad x \in \mathcal{S}. \tag{3.3}$$

In the sequel, we always denote by $t_{\text{opt}} \in [0, \pi/2]$ the optimal value of (3.3). Furthermore, for each fixed t, we denote by

$$\mathcal{S}(t) := \{\, x \in \mathcal{S} \mid u(x,t) \leq 0 \,\},$$

the set of points x that are feasible for (3.3), and by

$$\mathcal{S}^\circ(t) := \{\, x \in \mathcal{S}^\circ \mid u(x,t) < 0 \,\} \tag{3.4}$$

the interior of $\mathcal{S}(t)$. Note that $u(x,\pi/2) = -x_1$, and by (2.19) and (2.14), we have

$$\mathcal{S}(\pi/2) = \mathcal{S} \quad \text{and} \quad \mathcal{S}^\circ(\pi/2) \neq \emptyset. \tag{3.5}$$

Obviously, if $t = t_{\text{opt}}$, then equality must hold on both sides of (3.2). It follows that

$$\mathcal{S}(t_{\text{opt}}) = \{\, x \in \mathcal{S} \mid u(x, t_{\text{opt}}) = 0 \,\}, \quad \mathcal{S}^\circ(t_{\text{opt}}) = \emptyset, \tag{3.6}$$

and $\mathcal{S}(t_{\text{opt}})$ is just the set of optimal solutions of the convex fractional program (1.1). From (3.5) and (3.6), it follows that $0 \leq t_{\text{opt}} < \pi/2$. Finally, we remark that

$$\mathcal{S}^\circ(t) \neq \emptyset \quad \text{if} \quad t_{\text{opt}} < t \leq \pi/2, \quad \text{and} \quad \mathcal{S}(t) = \emptyset \quad \text{if} \quad 0 \leq t < t_{\text{opt}}. \tag{3.7}$$

In the following, let t with $t_{\text{opt}} < t \leq \pi/2$ be arbitrary, but fixed. Clearly, $\mathcal{S}(t)$ is a bounded convex set whose interior, by (3.7), is nonempty. We define a logarithmic barrier function $\varphi(\cdot, t)$ for $S^\circ(t)$ by setting

$$\varphi(x,t) := \phi(x) - m \log(-u(x,t)) \quad \text{for all} \quad x \in S^\circ(t). \tag{3.8}$$

Here, ϕ is the logarithmic barrier function for $\mathcal{S}$ given by (2.15). We stress that the factor m of the second term in (3.8) could be replaced by an arbitrary constant $\kappa > 0$. However, the choice $\kappa = m$ leads to the best constant in the convergence result presented in Theorem 4.3 in Section 4, and therefore, in this paper, we only consider the case $\kappa = m$.

Note that ϕ is assumed to be m-self-concordant. The function $-m \log(-u(\cdot,t))$ is obviously m-self-concordant. Therefore, in view of Lemma 2.1, the function $\varphi(\cdot,t)$ is strongly $(2m)$-self-concordant. Consequently, $\varphi(\cdot,t)$ satisfies all the properties listed in Section 2.1 (with $\mathcal{S}$ replaced by $\mathcal{S}(t)$ and m replaced by $2m$). In particular, recall from (2.4) that the analytic center—denoted by $x(t)$ in the sequel—of the set $\mathcal{S}(t)$ is the unique optimal solution of the unconstrained optimization problem

$$\underset{x \in \mathcal{S}(t)}{\text{minimize}} \ \varphi(x,t). \tag{3.9}$$

It can be shown that $x(t)$ converges to a point in $\mathcal{S}(t_{\text{opt}})$ as t approaches t_{opt}, and thus

$$x_{\text{opt}} := \lim_{t \to t_{\text{opt}}+0} x(t) \tag{3.10}$$

is an optimal solution of the original convex fractional program (1.1).

Finally, we note that, in view of (3.8) and (3.1), the gradient and the Hessian of $\varphi(\cdot,t)$ are given by

$$g(x,t) := \nabla_x \varphi(x,t) = \nabla\phi(x) - m\frac{v(t)}{u(x,t)}, \tag{3.11}$$

$$H(x,t) := \nabla_x^2 \varphi(x,t) = \nabla^2\phi(x) + m\frac{v(t)v(t)^T}{(u(x,t))^2}. \tag{3.12}$$

3.2. The method

The property (3.10) of the analytic centers $x(t)$ suggests the following procedure for computing an optimal solution of the convex fractional program (1.1). We choose a sequence

$$\{t_k\}_{k=0}^{\infty}, \quad \text{where} \quad t_{\text{opt}} < t_{k+1} < t_k \le \pi/2 \quad \text{for all} \quad k,$$

and we determine for each k an approximation—denoted by $x^{(k)}$ in the sequel—to the analytic center $x(t_k)$ of $\mathcal{S}(t_k)$. We remark that quantities corresponding to the kth iteration of this procedure are always marked by subscripts k, with the exception of the iterates $x^{(k)}$ for which superscripts (k) are used. This is done to avoid confusion with the notation x_1 and x_2 for the first and second component of a vector x.

If $\lim_{k\to\infty} t_k = t_{\text{opt}}$ and if we chose $x^{(k)} = x(t_k)$ then $x^{(k)}$ would converge to an optimal solution of (1.1). In our method, we use approximations $x^{(k)}$ that are obtained by a single Newton step at each kth iteration. We also propose a simple strategy for choosing the sequence $\{t_k\}_{k=0}^{\infty}$. As we will demonstrate in Section 4, this procedure still converges in a certain sense to the optimal solutions of the convex fractional program (1.1). More precisely, the rate of convergence is measured in terms of the area of two-dimensional convex sets $\mathcal{C}_k$ containing $x_{1,2}^{(k)}$ and the first two components of all optimal points of (1.1). We will show that the area of $\mathcal{C}_k$ is reduced by a constant factor at each iteration.

At the start of our method, we choose a t_0, $0 < t_0 \le \pi/2$, such that $\mathcal{S}^\circ(t_0) \ne \emptyset$. For example, by (3.5), one can set $t_0 = \pi/2$. Then, we determine a point $x^{(0)} \in \mathcal{S}^\circ(t_0)$ that is "close" to the analytic center of $\mathcal{S}(t_0)$ in the following sense:

$$\|g(x^{(0)}, t_0)\|_{(H(x^{(0)},t_0))^{-1}} \le \frac{1}{16}. \tag{3.13}$$

A point $x^{(0)}$ satisfying (3.13) can easily be obtained in polynomial time by using any of the algorithms for convex programming in [18, 11]. At each kth iteration, our algorithm decreases the current value t_k by setting $t_{k+1} = t_k - \Delta t_k$. Here the step size $\Delta t_k > 0$ is chosen suitably, based on the kth iterate $x^{(k)}$. Then, we perform one step of Newton's method for minimizing $\varphi(\cdot, t_{k+1})$ over $\mathcal{S}(t_{k+1})$ with the current iterate $x^{(k)}$ as starting point. The resulting Newton iterate is $x^{(k+1)}$. In Theorem 3.5 below, we will show that the iterates $x^{(k)}$ generated by this procedure remain close to the analytic center $x(t_k)$ in the sense that

$$\|g(x^{(k)}, t_k)\|_{(H(x^{(k)},t_k))^{-1}} \le \frac{1}{16} \quad \text{for all} \quad k = 0, 1, \ldots. \tag{3.14}$$

Note that, in (3.14), the kth gradient is measured in the $(H(x^{(k)}, t_k))^{-1}$-norm. Moreover, we remark that, in view of (3.9), $g(x(t_k), t_k) = 0$, and hence the left-hand side in (3.14) would be zero if $x^{(k)}$ were chosen as the analytic center $x(t_k)$ of $\mathcal{S}(t_k)$.

In the sequel, it will be convenient to use the following notation:

$$g_k := g(x^{(k)}, t_k), \quad \bar{g}_k := g(x^{(k)}, t_{k+1}), \tag{3.15}$$

$$H_k := H(x^{(k)}, t_k), \quad \bar{H}_k := H(x^{(k)}, t_{k+1}), \tag{3.16}$$

$$B_k := H_k^{-1}, \quad \bar{B}_k := \bar{H}_k^{-1}, \tag{3.17}$$

$$u_k := u(x^{(k)}, t_k), \quad \bar{u}_k := u(x^{(k)}, t_{k+1}). \tag{3.18}$$

The proposed algorithm can then be formulated as follows.

Algorithm 3.1. (Interior-Point Method for Convex Fractional Programs.)

0) Choose $t_0 \in (0, \pi/2]$ such that $\mathcal{S}^\circ(t_0) \neq \emptyset$.
Determine a point $x^{(0)} \in \mathcal{S}^\circ(t_0)$ such that

$$\|g_0\|_{B_0} \;\leq\; \frac{1}{16}.$$

For $n = 0, 1, \ldots,$ do :

1) Choose $\sigma_k > 0$ and set

$$\Delta t_k = \frac{\sigma_k |u_k|}{m\|x_{1,2}^{(k)}\|}. \tag{3.19}$$

2) Set

$$t_{k+1} = t_k - \Delta t_k. \tag{3.20}$$

3) Set

$$x^{(k+1)} = x^{(k)} - \bar{B}_k \bar{g}_k. \tag{3.21}$$

We remark that the quantities generated in Algorithm 3.1 are only defined if we can guarantee that $t_{\rm opt} < t_k$ and $x^{(k)} \in \mathcal{S}^\circ(t_k)$ for all k. As we will show in part a) of Theorem 3.5 in the next section, t_k and $x^{(k)}$ indeed satisfy these properties if $\sigma_k \leq 1/18$ for all k. Obviously, it is desirable to always choose the maximal value $\sigma_k = 1/18$ in (3.19) so that the sequence $\{t_k\}_{k=0}^{\infty}$ decreases as fast as possible. As our main result, namely Theorem 4.3 below, we will prove that Algorithm 3.1 converges in polynomial time if $\sigma_k = 1/18$ for all k.

3.3. Properties

In this subsection, we establish some basic properties of Algorithm 3.1. First, we prove two auxiliary results that will be needed later on.

Proposition 3.2. *Let $x \in \mathcal{S}^\circ$, and set $H_x = \nabla^2\phi(x)$. Then:*

$$\|h_{1,2}\| \;\leq\; \|x_{1,2}\| \quad \textit{for all} \quad h \in I\!R^n \quad \textit{with} \quad \|h\|_{H_x} \;\leq\; 1, \tag{3.22}$$

$$\|h\|_{H_x^{-1}} \;\leq\; \|h\| \cdot \|x_{1,2}\| \quad \textit{for all} \quad h = \begin{bmatrix} h_{1,2} \\ 0 \end{bmatrix} \in I\!R^n, \quad h_{1,2} \in I\!R^2. \tag{3.23}$$

Proof. Let $h \in I\!R^n$ with $\|h\|_{H_x} \leq 1$. From (2.3), it follows that $x \pm h \in \mathcal{S}$, and by (2.19), $x_1 \pm h_1 \geq 0$ and $x_2 \pm h_2 \geq 0$. Thus we have the estimate

$$\|h_{1,2}\|^2 = h_1^2 + h_2^2 \;\leq\; x_1^2 + x_2^2 = \|x_{1,2}\|^2,$$

which yields (3.22).

Letting M denote the leading principal 2×2 submatrix of H_x^{-1}, the inequality (3.23) is equivalent to the relation

$$h_{1,2}^T M h_{1,2} \;\leq\; \|h_{1,2}\|^2 \cdot \|x_{1,2}\|^2 \quad \text{for all} \quad h_{1,2} \in I\!R^2. \tag{3.24}$$

To show (3.24), note that, by (3.22), we have

$$H_x \;\geq\; \frac{1}{\|x_{1,2}\|^2}\,\mathrm{diag}(1, 1, 0, \ldots, 0). \tag{3.25}$$

It is easy to verify that (3.25) implies

$$M^{-1} \;\geq\; \frac{1}{\|x_{1,2}\|^2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{or, equivalently,} \quad M \;\leq\; \|x_{1,2}\|^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{3.26}$$

The second relation in (3.26) is equivalent to the inequality (3.24). □

Lemma 3.3. *Let $w, \Delta w \in I\!R^n$ and $\gamma > 0$ be given, and let M and $\bar{M}$ be the matrices defined by*

$$M := I + ww^T \quad \textit{and} \quad \bar{M} = I + \bar{w}\bar{w}^T, \quad \textit{where} \quad \bar{w} := \frac{1}{\gamma}(w + \Delta w), \tag{3.27}$$

and I is the $n \times n$ identity matrix. Then:

$$M \;\leq\; c\,\bar{M}, \quad \textit{where} \quad c := \max\{1, \gamma^2\} + \|\Delta w\|\,(\gamma + \|\Delta w\|)\,. \tag{3.28}$$

Proof. First, we remark that $M \leq c\bar{M}$ if, and only if,

$$\frac{y^T M y}{y^T \bar{M} y} \;\leq\; c \quad \text{for all} \quad y \in I\!R^n \quad \text{with} \quad \|y\| = 1. \tag{3.29}$$

Let $y \in I\!R^n$ with $\|y\| = 1$ be arbitrary, and set $\lambda := y^T\bar{w}$. With (3.27), it follows that

$$\begin{aligned}\frac{y^T M y}{y^T \bar{M} y} &= \frac{1 + (y^T w)^2}{1 + (y^T\bar{w})^2} = \frac{1 + (\gamma y^T\bar{w} - y^T\Delta w)^2}{1 + (y^T\bar{w})^2}\\ &= \frac{1 + \gamma^2\lambda^2}{1 + \lambda^2} + y^T\Delta w\left(\frac{y^T\Delta w}{1 + \lambda^2} - \frac{2\gamma\lambda}{1 + \lambda^2}\right).\end{aligned} \tag{3.30}$$

Next, we note that, for all $\lambda \in I\!R$, one has

$$\frac{1 + \gamma^2\lambda^2}{1 + \lambda^2} \;\leq\; \max\{1, \gamma^2\}, \quad \frac{1}{1 + \lambda^2} \;\leq\; 1, \quad \text{and} \quad \frac{2|\lambda|}{1 + \lambda^2} \;\leq\; 1. \tag{3.31}$$

Using (3.31) and the estimate $|y^T\Delta w| \leq \|\Delta w\|$, we obtain from (3.30) that

$$\frac{y^T M y}{y^T \bar{M} y} \;\leq\; \max\{1, \gamma^2\} + \|\Delta w\|\,(\gamma + \|\Delta w\|)\,. \tag{3.32}$$

In view of (3.28), the upper bound in (3.32) is just c, and thus (3.32) is the desired inequality (3.29). □

In the following theorem, we establish some first properties of the kth iterate $x^{(k)}$ of Algorithm 3.1.

Theorem 3.4. *Let $0 < t_k \leq \pi/2$, $x^{(k)} \in \mathcal{S}^\circ(t_k)$, and assume that*

$$\|g_k\|_{B_k} \;\leq\; \frac{1}{16}. \tag{3.33}$$

Let $\sigma_k > 0$, and let t_{k+1} be defined by (3.20) and (3.19). Then:

a)

$$\left|\frac{\bar{u}_k}{u_k} - 1\right| \le \frac{\sigma_k}{m}. \tag{3.34}$$

b) *If* $\sigma_k < m$, *then*:

$$t_{\text{opt}} < t_{k+1} < t_k \quad \textit{and} \quad x^{(k)} \in \mathcal{S}^\circ(t_{k+1}). \tag{3.35}$$

c) *If* $\sigma_k \le 1/18$, *then*:

$$\|\bar{g}_k\|_{\bar{B}_k} \le \frac{1}{5}. \tag{3.36}$$

Proof. For simplicity, we omit indices k and set $x = x^{(k)}$, $t = t_k$, $\Delta t = \Delta t_k$, $\sigma = \sigma_k$, $g = g_k$, $\bar{g} = \bar{g}_k$, and $\bar{B} = \bar{B}_k$. Furthermore, we set

$$v := v(t) \quad \text{and} \quad \bar{v} = v(t - \Delta t), \tag{3.37}$$

and we denote by $H_x = \nabla^2\phi(x)$ the Hessian of ϕ at x. Remark that, by (3.12), we have $0 < H_x \le \bar{H}_k$, and thus

$$\bar{B} = \bar{H}_k^{-1} \le H_x^{-1}. \tag{3.38}$$

Using (3.1), (3.37), and standard trigonometric identities, it follows that

$$v - \bar{v} = 2\sin(\Delta t/2)\dot{v}, \quad \text{where} \quad \dot{v} := \begin{bmatrix} -\cos(t - \Delta t/2) \\ -\sin(t - \Delta t/2) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in I\!R^n. \tag{3.39}$$

Note that $\|\dot{v}\| = 1$, and by (3.23) and (3.38), we have

$$\|\dot{v}\|_{\bar{B}} \le \|\dot{v}\|_{H_x^{-1}} \le \|x_{1,2}\|. \tag{3.40}$$

By multiplying (3.39) from the left by x^T and by using the estimate

$$2\sin(\Delta t/2) \le \Delta t, \tag{3.41}$$

we obtain

$$|u - \bar{u}| = 2\sin(\Delta t/2)|x^T\dot{v}| \le \Delta t\|x_{1,2}\|. \tag{3.42}$$

Combining (3.42) and (3.19) gives

$$|u - \bar{u}| \le \frac{\sigma|u|}{m}. \tag{3.43}$$

Note that, by assumption, $x \in \mathcal{S}^\circ(t)$, and thus, by (3.4), we have

$$|u| = -u > 0. \tag{3.44}$$

In particular, $|u| \ne 0$, and dividing (3.43) by $|u|$ gives (3.34). This concludes the proof of part a).

Next,we turn to part b), and we assume that $0 < \sigma < m$. Moreover, we set $\bar{t} := t_{k+1} = t - \Delta t$. From (3.43)–(3.44) and since $1 - \sigma/m > 0$, it follows that

$$\bar{u} \ \le u \left(1 - \frac{\sigma}{m}\right) \ < \ 0. \tag{3.45}$$

Note that, by (3.4), $x \in \mathcal{S}^\circ(t)$ implies that $x \in \mathcal{S}^\circ$. Thus, by (3.45) and (3.4), we conclude that $x \in \mathcal{S}^\circ(\bar{t})$, which is the second claim in (3.35). Clearly, $\bar{t} < t$, and therefore, it only remains to show that $t_{\text{opt}} < \bar{t}$. To this end, we first note that $\bar{u} = u(x, \bar{t})$ depends continuously on $\bar{t}$, and that, by (3.1), $u(x, 0) = x_2 > 0$. Since, by (3.45), $\bar{u} < 0$ for all $0 < \sigma < m$, we conclude that $\bar{t} > 0$. Suppose that $\bar{t} \le t_{\text{opt}}$. Recall from (3.6) and (3.7) that then $\mathcal{S}^\circ(\bar{t}) = \emptyset$, in contradiction to $x \in \mathcal{S}^\circ(\bar{t})$.

Finally, we turn to part c), and we assume that $0 < \sigma \le 1/18$. We will prove the following two inequalities:

$$\|g\|_{\bar{B}} \ \le \ \frac{1}{14}, \tag{3.46}$$

$$\|\bar{g} - g\|_{\bar{B}} \ \le \ \frac{9}{70} = \frac{1}{5} - \frac{1}{14}. \tag{3.47}$$

By the triangle inequality, (3.46) and (3.47) imply the desired estimate (3.36).

To show (3.46), we will derive the following more general result:

$$\bar{B} \ \le \ \frac{64}{49} B. \tag{3.48}$$

Clearly, from (3.48) and (3.33) it follows that

$$\|g\|_{\bar{B}} \ \le \ \sqrt{\frac{64}{49}} \|g\|_B \ \le \ \frac{8}{7} \cdot \frac{1}{16} = \frac{1}{14},$$

which is just (3.46). Since $\bar{B} > 0$, the claim (3.48) is equivalent to

$$B^{-1} \ \le \ \frac{64}{49} \bar{B}^{-1}. \tag{3.49}$$

Recall that the matrix H_x is symmetric positive definite, and hence it has a symmetric positive definite square root $H_x^{-1/2}$. By multiplying (3.49) from the left and the right by $H_x^{-1/2}$, we obtain the relation

$$M \ \le \ \frac{64}{49} \bar{M}, \quad \text{where} \quad M := H_x^{-1/2} B^{-1} H_x^{-1/2}, \quad \bar{M} := H_x^{-1/2} \bar{B}^{-1} H_x^{-1/2}. \tag{3.50}$$

Obviously, (3.50) and (3.49) are equivalent; we will prove (3.50). Using (3.12), (3.16)–(3.18), and (3.37), one easily verifies that the matrices M and $\bar{M}$ in (3.50) can be represented as follows:

$$\begin{aligned} M &= I + ww^T, \quad \text{where} \quad w := \frac{\sqrt{m}}{u} H_x^{-1/2} v, \\ \bar{M} &= I + \bar{w}\bar{w}^T, \quad \text{where} \quad \bar{w} := \frac{\sqrt{m}}{\bar{u}} H_x^{-1/2} \bar{v}. \end{aligned} \tag{3.51}$$

By (3.39), the vectors w and $\bar{w}$ satisfy

$$\bar{w} = \frac{1}{\gamma}(w + \Delta w), \quad \text{where} \quad \gamma := \frac{\bar{u}}{u}, \quad \Delta w := -\frac{\sqrt{m}}{u} 2\sin(\Delta t/2) H_x^{-1/2}\dot{v}.$$

From (3.34) and since $\sigma \le 1/18$, $m \ge 1$, we have

$$\gamma \;\le\; 1 + \frac{\sigma}{m} \;\le\; \frac{19}{18}. \tag{3.52}$$

Similarly, using (3.41), (3.19), and (3.40), we obtain that

$$\|\Delta w\| \;\le\; \frac{\sqrt{m}\,\Delta t}{|u|}\,\|\dot{v}\|_{H_x^{-1}} \;\le\; \frac{\sigma}{\sqrt{m}} \;\le\; \frac{1}{18}. \tag{3.53}$$

We now apply Lemma 3.3 to the matrices M and $\bar{M}$ in (3.51). Together with (3.52) and (3.53), it follows from (3.27) that

$$M \;\le\; c\,\bar{M}, \quad c \;\le\; \frac{19^2 + 19 + 1}{18^2} = \frac{127}{108} \;<\; \frac{64}{49},$$

which is just the claim (3.50).

It remains to prove the estimate (3.47). Note that, by (3.11) and (3.15), we have

$$\bar{g} - g = m\left(\frac{v}{u} - \frac{\bar{v}}{\bar{u}}\right) = \left(1 - \frac{u}{\bar{u}}\right)\frac{m}{u}v + \frac{m}{\bar{u}}(v - \bar{v}),$$

and thus

$$\|\bar{g} - g\|_{\bar{B}} \;\le\; \left|1 - \frac{u}{\bar{u}}\right| \frac{m}{|u|}\|v\|_{\bar{B}} + \frac{m}{|\bar{u}|}\|v - \bar{v}\|_{\bar{B}}. \tag{3.54}$$

From (3.11), we get

$$\frac{m}{u}v = \nabla\phi(x) - g. \tag{3.55}$$

Using (3.38) and the bound (2.11) of the length of the Newton step (2.5), we obtain

$$\|\nabla\phi(x)\|_{\bar{B}} \;\le\; \|\nabla\phi(x)\|_{H_x^{-1}} = \|H_x^{-1}\nabla\phi(x)\|_{H_x} \;\le\; \sqrt{m}. \tag{3.56}$$

By (3.55), (3.56), and (3.46), it follows that

$$\frac{m}{|u|}\|v\|_{\bar{B}} \;\le\; \sqrt{m} + \frac{1}{14}. \tag{3.57}$$

From (3.39)–(3.41) and (3.19), we get the estimate

$$\|v - \bar{v}\|_{\bar{B}} \;\le\; \Delta t\,\|x_{1,2}\| = \frac{\sigma|u|}{m}. \tag{3.58}$$

Next, we note that (3.43) implies

$$\frac{|u|}{|\bar{u}|} \;\le\; \frac{1}{1 - \sigma/m} \quad \text{and} \quad \left|1 - \frac{u}{\bar{u}}\right| \;\le\; \frac{\sigma/m}{1 - \sigma/m}. \tag{3.59}$$

Finally, by combining (3.54) and (3.57)–(3.59), we obtain

$$\|\bar{g} - g\|_{\bar{B}} \leq \frac{\sigma/m}{1-\sigma/m}\left(\sqrt{m} + \frac{1}{14} + m\right) = \frac{\sigma}{1-\sigma/m}\left(\frac{1}{\sqrt{m}} + \frac{1}{14m} + 1\right).$$

Since $m \geq 1$ and $\sigma \leq 1/18$, it follows that

$$\|\bar{g} - g\|_{\bar{B}} \leq \frac{\sigma}{1-\sigma}\left(1 + \frac{1}{14} + 1\right) \leq \frac{29}{17 \cdot 14} < \frac{9}{70},$$

which is the desired estimate (3.47). This concludes the proof of Theorem 3.4. □

We now show that, if $\sigma_k \leq 1/18$, then $t_{\text{opt}} < t_k$ and $x^{(k)} \in \mathcal{S}^\circ(t_k)$ for all k, and consequently, all quantities involved in Algorithm 3.1 are always well defined.

Theorem 3.5. *Let t_k and $x^{(k)}$ be the iterates generated by Algorithm 3.1, and assume that $0 < \sigma_k \leq 1/18$ for all k. Then, for all $k = 0, 1, \ldots,$ the following properties are satisfied:*

a)

$$t_{\text{opt}} < t_k \leq \pi/2 \quad \textit{and} \quad x^{(k)} \in \mathcal{S}^\circ(t_k), \tag{3.60}$$

$$\|g(x^{(k)}, t_k)\|_{(H(x^{(k)},t_k))^{-1}} \leq \frac{1}{16}. \tag{3.61}$$

b)

$$\|x^{(k)} - x(t_k)\|_{H(x^{(k)},t_k)} \leq \frac{1}{14}, \tag{3.62}$$

$$\|x^{(k)} - x(t_k)\|_{H(x(t_k),t_k)} \leq \frac{1}{13}. \tag{3.63}$$

Proof. We use induction on k to show part a). For $k = 0$, both properties (3.60) and (3.61) are satisfied. Indeed, the initial point $x^{(0)}$ of Algorithm 3.1 is chosen such that (3.61) holds. Furthermore, we have $0 < t_0 \leq \pi/2$, and thus, by part b) of Theorem 3.4 (with $k = 0$), $t_{\text{opt}} < t_0$. Now assume that (3.60) and (3.61) hold for some $k \geq 0$. We need to show that

$$t_{\text{opt}} < t_{k+1} \leq \pi/2, \tag{3.64}$$

$$x^{(k+1)} \in \mathcal{S}^\circ(t_{k+1}), \tag{3.65}$$

$$\|g_{k+1}\|_{B_{k+1}} \leq \frac{1}{16}. \tag{3.66}$$

Note that, by (3.60)–(3.61), the assumptions of Theorem 3.4 are fulfilled, and from (3.35), we immediately get (3.64). We now set $x := x^{(k)}$ and $\tilde{x} := x^{(k+1)}$. By the second part of (3.35), we have $x \in \mathcal{S}^\circ(t_{k+1})$. Recall that $\tilde{x} = x + \Delta x$ is obtained by performing one step—starting at x—of Newton's method (without line search) for minimizing $\varphi(\cdot, t_{k+1})$ over $\mathcal{S}(t_{k+1})$. Moreover, by (3.21) and since $\bar{B}_k = \bar{H}_k^{-1}$, we have

$$\Delta x = -\bar{B}_k \bar{g}_k = -\bar{H}_k^{-1} \bar{g}_k.$$

Together with part c) of Theorem 3.4, it follows that

$$\|\Delta x\|_{\bar{H}_k} = \|\bar{g}_k\|_{\bar{B}_k} \leq \frac{1}{5} < \frac{1}{4}. \tag{3.67}$$

Thus, we can apply the results (2.6) and (2.7) on Newton's method in Section 2.1.3 (with $\mathcal{S}$ replaced by $\mathcal{S}(t_{k+1})$ and $\phi(\cdot)$ replaced by $\varphi(\cdot, t_{k+1})$). From (2.6), we get $\tilde{x} \in \mathcal{S}^\circ(t_{k+1})$, which is just (3.65). Using (2.7) and (3.67), we obtain

$$\|\Delta\tilde{x}\|_{H_{\tilde{x}}} \leq \frac{\|\Delta x\|^2_{\bar{H}_k}}{(1-\|\Delta x\|_{\bar{H}_k})^2} \leq \frac{1/25}{(1-1/5)^2} = \frac{1}{16}. \tag{3.68}$$

Note that, in view of (2.9) and since $\tilde{x} = x^{(k+1)}$, we have

$$\|\Delta\tilde{x}\|_{H_{\tilde{x}}} = \|g_{k+1}\|_{B_{k+1}},$$

and thus (3.68) is equivalent to (3.66). This completes the proof of part a).

We now turn to part b). Let $k \geq 0$ be fixed, and set $x = x^{(k)}$, $t = t_k$, and $H = H(x^{(k)}, t_k)$. To prove (3.62), we make use of the property (2.8) of Newton's method in Section 2.1.3 (with $\mathcal{S}$ replaced by $\mathcal{S}(t)$ and $\phi(\cdot)$ replaced by $\varphi(\cdot, t)$). By (2.5) and since $B_k = H^{-1}$, the Newton step at x is given by

$$\Delta x = -H^{-1}\nabla_x\varphi(x,t) = -B_k g_k.$$

Thus, by (3.61), we have $\|\Delta x\|_H \leq 1/16 < 1/3$, and by applying (2.8) (with $x_\star = x(t)$), we get

$$\|x - x(t)\|_H \leq 1 - \left(1 - \frac{3}{16}\right)^{1/3} < \frac{1}{14}.$$

This is just the desired estimate (3.62).

Finally, the inequality (3.63) follows immediately from (3.62) by means of a general result on the norms induced by strongly self-concordant functions. Indeed, by applying [11, Lemma 2], we obtain

$$\|x - x(t)\|_{H(x(t),t)} \leq \frac{\|x - x(t)\|_H}{1 - \|x - x(t)\|_H} \leq \frac{1/14}{1 - 1/14} = \frac{1}{13}.$$

This completes the proof of Theorem 3.5. □

4. Convergence analysis

In this section, we present the main result of this paper, namely the polynomial-time convergence of Algorithm 3.1.

We continue to use the notation introduced in Section 3. In addition, for $t_{\text{opt}} \leq t \leq t_0$, we denote by

$$\mathcal{Q}(t) := \{\, z \in I\!R^2 \mid z = x_{1,2} \text{ for some } x \in \mathcal{S}(t)\,\} \tag{4.1}$$

the set in $I\!R^2$, which consists of all truncations $x_{1,2}$ of the vectors $x \in \mathcal{S}(t)$. We denote by $\mathcal{Q}^\circ(t)$ the interior of $\mathcal{Q}(t)$. Recall from (3.6) that

$$\mathcal{Q}(t_{\text{opt}}) = \{\, x_{1,2} \in I\!R^2 \mid x \text{ is an optimal solution of (1.1)}\,\} \tag{4.2}$$

just consists of the truncations of all optimal solutions of the convex fractional program (1.1). For all $k = 0, 1, \ldots$, we define sets

$$\mathcal{T}_k := \{\, z \in \mathbb{R}^2 \mid 0 \geq z^T v_{1,2}(t_k) \geq -d_k \,\}, \quad \text{where} \quad d_k := \frac{13}{6} \left| u(x^{(k)}, t_k) \right|. \tag{4.3}$$

Here, in view of (3.1), the vector $v_{1,2}(t)$ is given by

$$v_{1,2}(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}. \tag{4.4}$$

We remark that $\mathcal{T}_k$ is a stripe in $\mathbb{R}^2$ of width d_k. Finally, we set

$$\mathcal{D} := \left\{ z \in \mathbb{R}^2 \;\middle|\; z_1,\, z_2 \geq 0 \text{ and } \|z\| \leq \left(1 + \sqrt{2m}\right)^2 \|x_{1,2}(t_0)\| \right\}, \tag{4.5}$$

where $x(t_0)$ is the analytic center of $\mathcal{S}(t_0)$. Note that $\mathcal{D}$ is the quarter of a disk about the origin.

Both sets $\mathcal{Q}(t_k)$ and $\mathcal{T}_k$ contain the truncation $x_{1,2}^{(k)}$ of the kth iterate of Algorithm 3.1, as well as the set $\mathcal{Q}(t_{\text{opt}})$ of truncations of all optimal solutions of the convex fractional program (1.1). In the next theorem, we summarize these results.

Theorem 4.1. *Let t_k and $x^{(k)}$ be the iterates generated by Algorithm 3.1, and assume that $0 < \sigma_k \leq 1/18$ for all k. Then, for all $k = 0, 1, \ldots$, the following properties are satisfied:*

$$x_{1,2}^{(k)} \in \mathcal{Q}^\circ(t_k), \tag{4.6}$$

$$\mathcal{Q}(t_{\text{opt}}) \subseteq \mathcal{Q}(t_{k+1}) \subseteq \mathcal{Q}(t_k) \subseteq \mathcal{D}, \tag{4.7}$$

$$\mathcal{Q}(t_k) \subset \mathcal{T}_k. \tag{4.8}$$

For the proof of Theorem 4.1, we will need yet another property of the m-self-concordant logarithmic barrier function ϕ. Since this property is useful also in other situations, we prove the following result, which is somewhat more general than what is actually needed for Theorem 4.1.

Lemma 4.2. *Let $\mathcal{S} \in \mathbb{R}^n$ be the bounded convex set (1.4). Let $c : \mathcal{S} \mapsto \mathbb{R}$ be a continuously differentiable convex function such that the set*

$$\mathcal{S}_c := \{\, x \in \mathcal{S} \mid c(x) \leq 0 \,\}$$

has a nonempty interior $\mathcal{S}_c^\circ$. Let ϕ be the logarithmic barrier function (2.15) associated with $\mathcal{S}$, let $\kappa \geq 1$ be a constant, and let $\hat{x} \in \mathcal{S}_c$ be the unique optimal solution of

$$\underset{x \in \mathcal{S}_c}{\text{minimize}} \;\; (\phi(x) - \kappa \log(-c(x))). \tag{4.9}$$

Then:

$$\left(1 + \frac{m}{\kappa}\right) c(\hat{x}) \;\leq\; c(x) \;\leq\; 0 \quad \textit{for all} \quad x \in \mathcal{S}_c. \tag{4.10}$$

Proof. First, we recall that ϕ is strictly convex on the interior $\mathcal{S}^\circ$ of $\mathcal{S}$, and that $\phi = \infty$ on the boundary of $\mathcal{S}$. Clearly, the function in (4.9) has the same properties on the set $\mathcal{S}_c$, and this guarantees that (4.9) has a unique optimal solution $\hat{x}$ and $\hat{x} \in \mathcal{S}_c^\circ$. Let $x \in \mathcal{S}_c$ be arbitrary, but fixed, and set $h := x - \hat{x}$. Since c is convex, it follows that

$$c(\hat{x}) - c(x) = c(\hat{x}) - c(\hat{x} + h) \ \le\ -h^T \nabla c(\hat{x}). \tag{4.11}$$

We will prove that

$$\mu \ \le\ m, \quad \text{where} \quad \mu := \kappa \frac{-h^T \nabla c(\hat{x})}{-c(\hat{x})}. \tag{4.12}$$

Combining (4.11) and (4.12) then gives the desired result (4.10).

If $\mu \le 0$, then (4.12) is trivially satisfied. Assume that $\mu > 0$. Let $\psi(\beta) = \phi(\hat{x} + \beta h)$, $\psi : I \to I\!R$, where $I \supset [0,1)$ is an open interval such that $\hat{x} + \beta h \in \mathcal{S}_c^\circ$ for all $\beta \in I$. In view of (2.12), we have

$$\psi''(\beta) \ \ge\ \frac{(\psi'(\beta))^2}{m} \quad \text{for all} \quad \beta \in I. \tag{4.13}$$

Moreover, since $\hat{x}$ is the optimal solution of (4.9), it follows that $\psi'(0) = \mu$. Using the monotonicity of the differential inequality (4.13) (see, e.g., [15, Theorem 3.1]) and the observation that the function

$$\chi(\beta) := \left(\frac{1}{\mu} - \frac{\beta}{m}\right)^{-1}$$

satisfies $\chi(0) = \psi'(0)$ and $\chi'(\beta) = \chi^2(\beta)/m$, we conclude that

$$\psi'(\beta) \ \ge\ \chi(\beta).$$

Clearly, χ has its pole at $\beta = m/\mu > 0$, implying that ψ' and hence also ψ has a pole at some β_0 with $0 < \beta_0 \le m/\mu$. Since ψ has no pole in the interval $I \supset [0,1)$, it follows that $1 \le \beta_0 \le m/\mu$. This is just the claim (4.12). $\square$

Proof of Theorem 4.1. From the second relation in (3.60), we immediately get (4.6). We now turn to (4.7). First, we show that

$$\mathcal{Q}(t_0) \subseteq \mathcal{D}. \tag{4.14}$$

Let $z \in \mathcal{Q}(t_0)$ be arbitrary, but fixed. Since $x_1, x_2 \ge 0$ for all $x \in \mathcal{S}$, we have $z_1, z_2 \ge 0$. It remains to show the second condition in (4.5). We set $x := x(t_0)$ and $H_x := \nabla^2 \phi(x)$. Note that, in view of (3.12), we have $H_x \le H(x, t_0)$. Using (2.13) (applied to the $(2m)$-self-concordant function $\varphi(\cdot, t_0)$), we conclude that

$$z = x_{1,2} + 2\left(m + \sqrt{2m}\right) h_{1,2} \quad \text{for some} \quad h \in I\!R^n \quad \text{with} \quad \|h\|_{H_x} \ \le\ 1. \tag{4.15}$$

From (4.15) and (3.22), it follows that

$$\|z\| \ \le\ \left(1 + 2(m + \sqrt{2m})\right) \|x_{1,2}\| = \left(1 + \sqrt{2m}\right)^2 \|x_{1,2}\|,$$

and thus $z \in \mathcal{D}$.

Next, we note that $v_{1,2}(t)$ given by (4.4) clearly satisfies

$$z^T v_{1,2}(t') \ \ge\ z^T v_{1,2}(t) \quad \text{for all} \quad 0 \le t' < t \le \frac{\pi}{2} \quad \text{and} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z_1, z_2 \ge 0.$$

Consequently, we have $\mathcal{Q}(t') \subseteq \mathcal{Q}(t)$ for all $0 \leq t' < t \leq \pi/2$. Since t_k is monotonously decreasing, this—along with (4.14)—implies (4.7).

We now turn to (4.8). By the definitions of $\mathcal{Q}(t)$ and $\mathcal{T}_k$ in (4.1) and (4.3), and since $|u(x^{(k)}, t_k)| = -(x^{(k)})^T v(t_k)$, the inclusion (4.8) is equivalent to the following estimates:

$$0 \ \geq \ x^T v(t_k) \ \geq \ \frac{13}{6} (x^{(k)})^T v(t_k) \quad \text{for all} \quad x \in \mathcal{S}(t_k). \tag{4.16}$$

In order to prove (4.16), we apply Lemma 4.2 with $c(x) := x^T v(t_k)$, $\mathcal{S}_c := \mathcal{S}(t_k)$, and $\kappa := m$. By (3.9) and (3.8), the unique optimal solution $\hat{x}$ of (4.9) is just the analytic center $x(t_k)$ of $\mathcal{S}(t_k)$, and from (4.10), we obtain that

$$0 \ \geq \ x^T v(t_k) \ \geq \ 2\,(x(t_k))^T v(t_k) \quad \text{for all} \quad x \in \mathcal{S}(t_k). \tag{4.17}$$

It only remains to show that

$$(x(t_k))^T v(t_k) \ \geq \ \frac{13}{12} (x^{(k)})^T v(t_k), \tag{4.18}$$

since (4.17) and (4.18) imply the desired estimate (4.16). To this end, note that, by inequality (3.63) from Theorem 3.5, the vector

$$h := 13 \left(x^{(k)} - x(t_k) \right) \tag{4.19}$$

satisfies $\|h\|_{H(x(t_k),t_k)} \leq 1$. Thus, by the inner ellipsoidal approximation in (2.3) (with $\mathcal{S}$ replaced by $\mathcal{S}(t_k)$), we have

$$x(t_k) + h \in \mathcal{S}(t_k).$$

Together with (4.19), it follows that

$$0 \ \geq \ (x(t_k) + h)^T \ v(t_k) = 13\,(x^{(k)})^T v(t_k) - 12\,(x(t_k))^T v(t_k).$$

This inequality is equivalent to (4.18). □

After these preparations, we are now in a position to prove our convergence result for Algorithm 3.1. We set

$$\mathcal{C}_k := \mathcal{D} \cap \mathcal{T}_0 \cap \mathcal{T}_1 \cap \cdots \cap \mathcal{T}_k,$$

where $\mathcal{D}$ and $\mathcal{T}_j$, $j = 0, 1, \ldots, k$, are the sets defined in (4.5) and (4.3). Furthermore, for compact sets $\mathcal{C} \subset \mathbb{R}^2$, we denote by $\nu(\mathcal{C}) := \text{area}(\mathcal{C})$ the area of $\mathcal{C}$. Our convergence result can then be formulated as follows.

Theorem 4.3. *Let $x^{(k)}$ be the kth iterate generated by Algorithm* 3.1 *with $\sigma_k = 1/18$ for all k. Let m be the self-concordance parameter of the logarithmic barrier function* (2.15) *associated with the feasible set $\mathcal{S}$ of the convex fractional program* (1.1)*. Then, for all $k = 0, 1, \ldots,$ the following properties are satisfied:*

$$x_{1,2}^{(k)} \in \mathcal{C}_k^{\circ}, \tag{4.20}$$

$$x_{1,2} \in \mathcal{C}_k \quad \textit{for all optimal solutions } x \textit{ of } (1.1), \tag{4.21}$$

$$\nu(\mathcal{C}_{k+1}) \ \leq \ \left(1 - \frac{1}{80\,m}\right) \nu(\mathcal{C}_k). \tag{4.22}$$

Proof. The properties (4.20) and (4.21) follow immediately from the relations (4.6)–(4.8) in Theorem 4.1 and from (4.2).

We now turn to (4.22). Note that, in view of (4.3), each stripe $\mathcal{T}_j$ is bounded by two parallel lines with slope $\tan t_j$. Moreover, the left line passes through the origin 0 in $\mathbb{R}^2$. Recall that $0 < t_{j+1} < t_j \leq \pi/2$ for all j. It follows that the upper left boundary of the set $\mathcal{C}_k$ consists of a segment of the form $[0, p_k]$ of the left borderline of $\mathcal{T}_k$. Here, p_k is given by

$$p_k := \rho_k \begin{bmatrix} \cos t_k \\ \sin t_k \end{bmatrix}, \quad \text{where} \quad \rho_k := \max \left\{ \rho \in \mathbb{R} \;\middle|\; \rho \begin{bmatrix} \cos t_k \\ \sin t_k \end{bmatrix} \in \mathcal{C}(t_k) \right\}.$$

Let p_{k+1} be the corresponding boundary point of $\mathcal{C}_{k+1}$, and let $\mathcal{F}_k$ denote the triangle with vertices 0, p_k, and p_{k+1}. Note that

$$\nu(\mathcal{F}_k) = \frac{1}{2} \rho_k \rho_{k+1} \sin \Delta t_k, \tag{4.23}$$

where, by (3.20) and (3.19) (with $\sigma_k = 1/18$),

$$\Delta t_k = t_k - t_{k+1} = \frac{|u(x^{(k)}, t_k)|}{18\, m\, \|x_{1,2}^{(k)}\|}. \tag{4.24}$$

By construction of the triangle $\mathcal{F}_k$, we have

$$\mathcal{F}_k \subset \mathcal{C}_k \quad \text{and} \quad \mathcal{C}_{k+1} \subset \mathcal{C}_k \setminus \mathcal{F}_k^\circ,$$

and thus

$$\nu(\mathcal{C}_{k+1}) \;\leq\; \nu(\mathcal{C}_k) - \nu(\mathcal{F}_k) = \left(1 - \frac{\nu(\mathcal{F}_k)}{\nu(\mathcal{C}_k)}\right) \nu(\mathcal{C}_k). \tag{4.25}$$

Next, recall the definition of d_k in (4.3), and set

$$q_k := d_k \begin{bmatrix} \sin t_k \\ -\cos t_k \end{bmatrix}, \quad \text{where} \quad d_k = \frac{13}{6} |u(x^{(k)}, t_k)|. \tag{4.26}$$

Observe that, by construction of $\mathcal{C}_k$ as the intersection of the disk $\mathcal{D}$ and the stripes $\mathcal{T}_j$, $j = 0, 1, \ldots, k$, the angle of the set $\mathcal{C}_k$ at the vertex p_k is at most $\pi/2$, and hence

$$\mathcal{C}_k \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; z^T \begin{bmatrix} \cos t_k \\ \sin t_k \end{bmatrix} \leq \rho_k \right\}. \tag{4.27}$$

Furthermore, the radius ρ_k of the vertex p_k is nonincreasing, i.e., $\rho_j \geq \rho_{j+1}$ for all $j = 0, 1, \ldots$, implying that

$$\mathcal{C}_k \subset \left\{ z \in \mathbb{R}^2 \;\middle|\; \|z\| \leq \rho_k \right\}. \tag{4.28}$$

In view of (4.27) and since $\mathcal{C}_k \subset \mathcal{T}_k$, the set $\mathcal{C}_k$ is contained in the rectangle $\mathcal{R}_k$ whose vertices are the points 0, p_k, $p_k + q_k$, and q_k. Moreover, one easily verifies that p_{k+1} is a boundary point of $\mathcal{C}_k$, and that

$$\mathcal{C}_k \subset \mathcal{R}_k \setminus (\Delta R_k)^\circ \quad \text{and} \quad \Delta R_k \subset \mathcal{R}_k. \tag{4.29}$$

Here, ΔR_k is the rectangle with vertices p_{k+1}, $p_k + s_k q_k$, $p_k + q_k$, and $c_k p_k + q_k$, and

$$s_k = \rho_{k+1} \sin \Delta t_k \quad \text{and} \quad c_k = \rho_{k+1} \cos \Delta t_k. \tag{4.30}$$

From (4.29) and (4.30), it follows that

$$\begin{aligned}\nu(\mathcal{C}_k) &\leq \nu(\mathcal{R}_k) - \nu(\Delta R_k) \\ &= \rho_k d_k - (\rho_k - \rho_{k+1}\cos\Delta t_k)(d_k - \rho_{k+1}\sin\Delta t_k) \\ &\leq \rho_k\rho_{k+1}\sin\Delta t_k + d_k\rho_{k+1}\cos\Delta t_k.\end{aligned} \tag{4.31}$$

Note that, by (4.20) and (4.28), we have $\|x_{1,2}^{(k)}\| \leq \rho_k$. Together with (4.24) and (4.26), it follows that

$$\tan\Delta t_k \;\geq\; \Delta t_k = \frac{6\ d_k}{13\cdot 18\, m\, \|x_{1,2}^{(k)}\|} \;\geq\; \frac{d_k}{39\, m\, \rho_k}. \tag{4.32}$$

From (4.31), (4.23), and (4.32), we obtain

$$\frac{\nu(\mathcal{C}_k)}{\nu(\mathcal{F}_k)} \;\leq\; 2\left(1 + \frac{d_k}{\rho_k \tan\Delta t_k}\right) \;\leq\; 2\,(1 + 39\, m) \;\leq\; 80\, m.$$

In combination with (4.25), this implies the desired estimate (4.22). □

Note that $\mathcal{C}_0 \subseteq \mathcal{D}$, and thus, by (4.5), we have

$$\nu(\mathcal{C}_0) \;\leq\; \rho_0^2\,\frac{\pi}{4}, \quad \text{where} \quad \rho_0 = \left(1 + \sqrt{2m}\right)^2 \|x_{1,2}(t_0)\|.$$

Together with (4.22), it follows that

$$\nu(\mathcal{C}_k) \;\leq\; \left(1 - \frac{1}{80\, m}\right)^k \rho_0^2\,\frac{\pi}{4} \quad \text{for all} \quad k = 0, 1, \ldots\,.$$

Therefore, the area $\nu(\mathcal{C}_k)$, which measures the convergence of Algorithm 3.1, converges to zero in polynomial time.

5. Stopping criterion

In this section, we discuss optimality conditions for convex fractional programs, and we propose a practical stopping criterion.

5.1. The KKT conditions

We now denote by $f_0(x) := x_2/x_1$ the objective function of the convex fractional program (1.1). If there are feasible points $x \in \mathcal{S}$ with $x_1 = 0$, then, in view of (2.19), $f_0(x) = \infty$ at any such point x. Consequently, all optimal solutions of (1.1) are contained in the set

$$\tilde{\mathcal{S}} := \{\, x \in \mathcal{S} \mid x_1 > 0 \,\},$$

and thus, we only need to discuss optimality conditions for points $x \in \tilde{\mathcal{S}}$. We remark that, with $\mathcal{S}$, the set $\tilde{\mathcal{S}}$ is also bounded and convex.

The objective function f_0 of (1.1) is *pseudo-convex* [16] on $\tilde{\mathcal{S}}$, i.e., for all $x \in \tilde{\mathcal{S}}$, f_0 is differentiable at x and

$$(\nabla f_0(x))^T(\tilde{x} - x) \geq 0, \quad \tilde{x} \in \tilde{\mathcal{S}} \quad\Longrightarrow\quad f_0(\tilde{x}) \geq f_0(x).$$

The pseudo-convexity of f_0 guarantees that the *Karush-Kuhn-Tucker conditions* (referred to as KKT conditions in the sequel) are necessary and sufficient for optimality; see, e.g., Mangasarian [16]. The KKT conditions for (1.1) are satisfied if, and only if, one can find a point $x \in I\!R^n$ and a vector $y \in I\!R^s$ such that

$$\nabla f_0(x) + \sum_{i=1}^{s} y_i \nabla f_i(x) = 0 \tag{5.1}$$

and, for all $i = 1, 2, \ldots, s$,

$$f_i(x) \;\leq\; 0, \quad y_i \;\geq\; 0, \tag{5.2}$$

$$y_i\, f_i(x) \;=\; 0. \tag{5.3}$$

5.2. A practical stopping criterion

At central points the KKT conditions are nearly satisfied in the following sense. Let $x = x(t)$ be the analytic center of the set $\mathcal{S}(t)$. Let the vector $y \in I\!R^s$ be defined by

$$y_i := \frac{u(x,t)}{m\, x_1^2\, f_i(x)} \left(x_1 \cos t + x_2 \sin t\right), \quad i = 1, 2, \ldots, s. \tag{5.4}$$

By definition of the center x, we have $\nabla_x \varphi(x,t) = 0$ or, equivalently,

$$-\sum_{i=1}^{s} \frac{\nabla f_i(x)}{f_i(x)} = \frac{m\, v(t)}{u(x,t)}. \tag{5.5}$$

Moreover, note that

$$\nabla f_0(x) = \frac{1}{x_1^2} \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \|\nabla f_0(x)\| = \frac{\|x_{1,2}\|}{x_1^2}. \tag{5.6}$$

We now define the *relative residual vector* of the KKT equation (5.1) as follows:

$$r(x,t) := \frac{1}{\|\nabla f_0(x)\|} \left(\nabla f_0(x) + \sum_{i=1}^{s} y_i \nabla f_i(x) \right). \tag{5.7}$$

Using (5.4)–(5.7), we get

$$\|r(x,t)\| = \frac{1}{\|x_{1,2}\|} \left\| \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} - (x_1 \cos t + x_2 \sin t) \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \right\| = \frac{|u(x,t)|}{\|x_{1,2}\|}. \tag{5.8}$$

By definition, $x \in \mathcal{S}^\circ$, and therefore, we have $f_i(x) < 0$ and $y_i > 0$. Hence the inequalities (5.2) are always fulfilled. It turns out that a suitably normalized version of the complementary slackness condition (5.3) is also satisfied up to a relative error of size $\|r(x,t)\|$. Using the normalization factor $1/(\|x_{1,2}\| \cdot \|\nabla f_0(x)\|)$, we obtain

$$\frac{|y_i\, f_i(x)|}{\|x_{1,2}\| \cdot \|\nabla f_0(x)\|} = \frac{|u(x,t)|}{m} \cdot \frac{x_1 \cos t + x_2 \sin t}{\|x_{1,2}\|} \;\leq\; \frac{\|r(x,t)\|}{m}. \tag{5.9}$$

We stress that this normalization is indeed natural for our problem: The normalized complementary slackness condition (5.9) is invariant under scaling of f_0 by a positive constant (i.e., for a given x, condition (5.9) is the same for the problems "minimize $f_0(x)$" and "minimize $\lambda f_0(x)$" for any fixed $\lambda > 0$), while the condition $y_i f_i \leq \epsilon$ derived from (5.2) is not invariant. Similarly, (5.9) is also invariant under scaling of x.

Since the kth iterate $x^{(k)}$ is close to the analytic center $x(t_k)$ of $\mathcal{S}(t_k)$, the relations (5.8) and (5.9) suggest the following practical stopping criterion:

$$\frac{|u(x^{(k)}, t_k)|}{\|x_{1,2}^{(k)}\|} \leq \texttt{tol}, \tag{5.10}$$

where `tol` is some convergence tolerance.

6. A sketch of a practical implementation

In an actual implementation of our interior-point method, we include a predictor step. Before outlining the resulting implementation, we briefly motivate the predictor step.

6.1. The predictor step

By the implicit function theorem, $x(t)$ is a smooth curve in the variable $t > t_{\text{opt}}$. Furthermore, in view of (3.10), $x(t)$ approaches an optimal solution as $t \to t_{\text{opt}}$. Fiacco and McCormick [6] showed that these properties also hold for the curves $\tilde{x}(t)$ where $\tilde{x}(t) = \arg\min_x \varphi(x,t) + w^T x$ with some constant linear perturbation $w^T x$.

Instead of computing $x^{(k+1)}$ by Newton's method starting from $x^{(k)}$, this suggests to use

$$\tilde{x}^{(k+1)} := x^{(k)} - \tilde{x}'(t_k)\Delta t_k \tag{6.1}$$

as a starting vector for Newton's method. Here, the linear perturbation w defining $\tilde{x}$ is chosen as $w = g(x^{(k)}, t_k)$ and depends on k. (It is chosen such that $\tilde{x}(t_k) = x^{(k)}$!) Recall from (3.11) that $g(x,t) = \nabla_x \varphi(x,t)$ denotes the gradient of $\varphi(\cdot, t)$. In order to be able to use (6.1), we need an approximation to the tangent vector $\tilde{x}'(t_k)$ of the curve $\tilde{x}(t)$ at $t = t_k$.

Note that, since $\tilde{x}(t)$ minimizes $\varphi(x,t) + w^T x$, and by (3.11), we have

$$-w = g(\tilde{x}(t), t) = \nabla \phi(\tilde{x}(t)) - m \frac{v(t)}{u(\tilde{x}(t), t)} \quad \text{for all} \quad t. \tag{6.2}$$

Differentiating (6.2) with respect to t, it follows that

$$0 = \nabla_x^2 \phi(\tilde{x}(t)) \tilde{x}'(t) - m \frac{v'(t)}{u(\tilde{x}(t), t)} + m \frac{v(t)}{(u(\tilde{x}(t), t))^2} \left((\tilde{x}(t))^T v'(t) + (\tilde{x}'(t))^T v(t) \right) \tag{6.3}$$

for all t. For $t = t_k$, the relation (6.3) can be rewritten in the form

$$\left(\nabla_x^2 \phi(x^{(k)}) + m \frac{v(t_k) v(t_k)^T}{(u(x^{(k)}, t_k))^2} \right) \tilde{x}'(t_k) = m \frac{v'(t_k)}{u(x^{(k)}, t_k)} - m \frac{v(t_k) \left(v'(t_k)^T x^{(k)} \right)}{(u(x^{(k)}, t_k))^2}. \tag{6.4}$$

Hence we can compute $\tilde{x}'(t_k)$ by solving the nonsingular system of linear equations (6.4).

6.2. Implementable algorithm

An outline of our implementation is then as follows:

- Let an initial point $x^{(0)} \in \mathcal{S}^o$ be given.
- Determine $t = t_0 \in (0, \pi/2]$ such that $\|g(x^{(0)}, t)\|$ is minimal.
- Use Newton's method to minimize $\varphi(x, t_0)$ and let the result be $x^{(1)}$. Set $k = 1$ and $t_1 = t_0$.

As long as the stopping criterion is not satisfied repeat the following steps:

- Compute $\tilde{x}'(t_k)$ by solving the linear system (6.4).
- Find $\Delta t_{\max} = \max\{\, \Delta t \in I\!R \mid x^{(k)} - \Delta t\, \tilde{x}'(t_k) \in \mathcal{S} \,\}$.
- Choose a reduction factor $\tau_k \in (0, 1)$ and set $\Delta t_k = \tau_k\, \Delta t_{\max}$.
- Extrapolate $\tilde{x}^{(k+1)} := x^{(k)} - \Delta t_k\, \tilde{x}'(t_k)$.
- Find $t_{k+1} < t_k$ (e.g., by minimizing $\|g(\tilde{x}^{(k+1)}, t)\|$).
- Use Newton's method with line search and starting point $\tilde{x}^{(k+1)}$ to minimize $\varphi(x, t_{k+1})$. Let the result be $x^{(k+1)}$.
- Set $k = k + 1$.

Technical details, such as the choice of τ_k, the choice of $t_{k+1} < t_k$ such that a sufficient reduction in t can be guaranteed, or how to choose the stopping criterion for Newton's method at each step of the loop, are omitted here (they are as described in [13]).

The extrapolation step is also called "affine scaling step" in the literature (for linear and convex programming), and Newton's method bringing the extrapolated point closer to the point $x(t_{k+1})$ on the central curve is called corrector step or centering step.

Finally, we consider the special case (1.12) of minimizing the condition number of positive definite matrices $M(x)$. Here, given a starting point $x^{(0)}$ such that $M(x^{(0)}) > 0$, we initialized our program by setting $\alpha = \lambda_{\min}(M(x))/2$ and $\beta = \lambda_{\max}(M(x^{(0)}) + \alpha$. For this problem, minimizing the norm of the gradient can be done by a simple formula: Setting $p := \operatorname{trace}((M(x) - \alpha I)^{-1})$ and $q := \operatorname{trace}((\beta I - M(x))^{-1})$, we obtain

$$t = \arctan\left(\frac{p\beta^2 + q\alpha\beta + \alpha}{p\alpha\beta - \beta + q\alpha^2}\right).$$

7. Numerical experiments

The algorithm sketched in the previous section has been implemented in MATLAB™. In this section, we present results of two numerical experiments to give some insight into the practical behavior of our method. The first example examines a simple modification of the SSOR preconditioner for a matrix A derived from an elliptic partial differential equation. The second example determines a completion with a minimal condition number of a nonsymmetric block matrix A with a prescribed zero structure. For both examples our algorithm shows the same convergence behavior—a very slow growth of the number of iterations with increasing dimension—as it has also been observed in the case of interior-point methods for solving linear and convex programs.

In all runs, we used the stopping criterion (5.10), i.e., the algorithm was terminated as soon as

$$\varrho_k := \frac{|u(x^{(k)}, t_k)|}{\|x_{1,2}^{(k)}\|} \leq \texttt{tol} \tag{7.1}$$

was reached. All our examples were run on an SGI with machine precision of order $\mathcal{O}(10^{-16})$.

7.1. Example 1

The following example allows us to compute preconditioners M for a positive definite matrix A such that the condition number of $M^{-1}A$ is minimal. Our method is certainly not practical if all we want is to solve a system $Ax = b$. However, it allows us to compute "optimal preconditioners" for small matrices A—and to look at them and to derive heuristics for larger matrices A.

Let A be a symmetric positive definite matrix. We assume that A has been scaled such that all its diagonal elements are equal to one. Then, denoting by L the strictly lower triangular part of A, we have the splitting

$$A = L + I + L^T. \tag{7.2}$$

of the matrix A. The splitting (7.2) suggest to consider preconditioning matrices M of the form

$$M = M(D) = (I + L)(I + D)(I + L^T),$$

where D is a diagonal matrix with $D > -I$. For $D = 0$, we obtain the SSOR preconditioner [2] with relaxation parameter $\omega = 1$. Our goal is to find the diagonal matrix D such that the Euclidean condition number of $(M(D))^{-1}A$ is minimized. To this end, let $A = R^T R$, where R is upper triangular, be the Cholesky decomposition of A. Using

$$\begin{aligned} \operatorname{cond}\left((M(D))^{-1}A\right) &= \operatorname{cond}\left((M(D))A^{-1}\right) \\ &= \operatorname{cond}\left((M(D))R^{-1}R^{-T}\right) = \operatorname{cond}\left(R^{-T}(M(D))R^{-1}\right), \end{aligned}$$

we minimize the condition of $R^{-T}(M(D))R^{-1}$. Observe that the unknown variable D appears linearly in $R^{-T}(M(D))R^{-1}$. Thus, the resulting problem is of the type (1.10), and we can use the approach described in Section 1.2 for its solution.

In our examples below we considered matrices A resulting from a finite difference approximation of the Laplacian on an $n \times n$ grid. Note that the number of unknowns is of order n^2. In Table 1, we list the total number of iterations and Newton steps of our algorithm, as well as the optimal condition number and the condition number of the SSOR preconditioner, for the cases $n = 4,\ 6,\ 8$, and 10. In all four runs of this example, the convergence tolerance `tol = 1.0e-06` was used for the stopping test (7.1). It is worth noticing that the condition number of the SSOR preconditioner is almost optimal in all four cases.

For the case $n = 10$, the iteration number k, the value of ϱ_k from (7.1) at iteration k, the value of $\tan t_k$, and the number of Newton steps needed at iteration k are listed in Table 2.

Problem	n = 4	n = 6	n = 8	n = 10
Iterations	5	5	5	6
Newton steps	27	33	39	52
Optimal condition	1.939	3.160	4.788	6.820
SSOR condition	2.008	3.227	4.854	6.886

Table 1: Iterations counts and optimal and SSOR condition numbers for Example 1

Iteration	ϱ_k	$\tan t_k$	Newton steps
k=0	5.119e-02	14.96	13
k=1	1.024e-02	14.96	7
k=2	1.951e-03	7.565	6
k=3	2.309e-04	6.939	6
k=4	1.857e-05	6.833	6
k=5	1.033e-06	6.821	8
k=6	4.439e-08	6.820	6

Table 2: Convergence history for Example 1 with $n = 10$

7.2. Example 2

Here we consider matrices A of the form

$$A = \begin{bmatrix} \bar{A} & \bar{B} \\ 0 & X \end{bmatrix}$$

with given blocks $\bar{A} \in \mathbb{R}^{n\times l}$ and $\bar{B} \in \mathbb{R}^{n\times m}$. The problem is to find a matrix $X \in \mathbb{R}^{(l+m-n)\times m}$ that minimizes the Euclidean condition number of A. If $l + m - n \leq m$, the problem can be transformed into the equivalent problem of minimizing the condition number of the matrix

$$M = A^T A = \begin{bmatrix} \bar{A}^T\bar{A} & \bar{A}^T\bar{B} \\ \bar{B}^T\bar{A} & \bar{B}^T\bar{B} + X^T X \end{bmatrix}.$$

Since $l + m - n \leq m$, we can substitute $X^T X$ by another positive definite matrix Z and minimize the condition number of M with respect to Z. Note that M depends affinely on Z, and again, the approach described in Section 1.2 can be used to minimize the condition number of M. By taking square roots, we then obtain an optimal solution for the original problem.

In our example below, we chose an upper triangular matrix $\bar{A}$ of size $l \times l$ with random entries and a random matrix $\bar{B}$ of size $l \times m$. The entries of $\bar{A}$ and $\bar{B}$ are uniformly distributed in $(0, 1)$. As initial matrix Z such that M is positive definite, we chose $Z = (\|\bar{A}^T\bar{A}\| + \|\bar{A}^T\bar{B}\|)I$. In Table 3, we list the number of iterations and Newton steps for the five problems with $m = 2,\ 4,\ 6,\ 8$, and 10, and fixed $l = 10$. In all five runs of this example, the convergence tolerance `tol = 1.0e-08` was used for the stopping test (7.1).

Problem	m = 2	m = 4	m = 6	m = 8	m = 10
Iterations	3	2	2	2	2
Newton steps	29	31	33	37	39

Table 3: Number of iterations and Newton steps for Example 2

8. Concluding remarks

We presented an interior-point method for solving convex fractional programs. As our main result, we showed that the method converges in polynomial time. Here, convergence is measured in terms of suitably constructed two-dimensional sets that contain the leading two components of the iterates generated by our method, as well as the leading two components of all optimal solutions of the convex fractional program. We stress that the class of problems considered in this paper is not convex, but only pseudo-convex, and therefore, the existence of a polynomial-time method is somewhat surprising. We also discussed a practical implementation of the proposed method, and we presented results of a few numerical experiments. These results suggest that the proposed method is a viable approach for solving problems arising in practical applications.

Finally, we mention two possible extensions of the method proposed in this paper. First, the class of convex fractional programs considered here is the special case $l = 1$ of the more general problem of minimizing the maximum of l fractions over some convex set. Techniques similar to the ones in this paper can be employed to devise and analyze an interior-point method for this more general problem. Such a method will be described in detail in the forthcoming paper [7]. Second, by introducing an additional artificial constraint, it is possible to construct a theoretical interior-point method for convex fractional programs whose convergence rate depends on $1/\sqrt{m}$, instead of $1/m$. In [8], we use such an interior-point method to derive results on the complexity of convex fractional programs. We stress that this method is only of theoretical interest, and the additional artificial constraint does not lead to a better practical algorithm. This is the reason why we did not include this additional constraint in the analysis presented in this paper.

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