A Polynomial-Time Algorithm for Fractional Programs with Convex Constraints

Roland W. Freund
AT&T Bell Laboratories
600 Mountain Avenue, Room 2C–420
Murray Hill, New Jersey 07974–0636, U.S.A.

Florian Jarre
Institut für Angewandte Mathematik und Statistik
Universität Würzburg
Am Hubland
D–8700 Würzburg, Federal Republic of Germany

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Abstract. We describe an interior-point method for solving a class of fractional programs. Even though the programs under consideration are only pseudo-convex, the proposed method converges in polynomial time, just as in the case of a convex problem. Here, the rate of convergence is measured in terms of suitably constructed two-dimensional sets whose areas are reduced by a constant factor at each iteration.

1 Introduction

Karmarkar's 1984 landmark paper [6] about a practical polynomial-time interior-point algorithm for linear programs triggered extensive research efforts into interior-point techniques for nonlinear optimization problems, and since 1984, numerous new algorithms have been proposed. However, in contrast to linear programs, interior-point methods for nonlinear programs converge in polynomial time only in special cases, such as classes of convex programming problems; we refer the reader to [7, 5, 1, 2] and the references given there.

In this note, we describe a polynomial-time interior-point algorithm for solving fractional programs of the following standard form

$$\begin{aligned} &\text{minimize}_{x\in I\!R^n} \ \tfrac{x_2}{x_1} \\ &\text{subject to} \quad x \in \mathcal{S} := \{\, x \in I\!R^n \mid f_i(x) \le 0,\ i = 1, 2, \ldots, s \,\}. \end{aligned} \tag{1}$$

Here, $n \ge 2$, and x_1 and x_2 denote the first two components of $x \in I\!R^n$. Furthermore, $f_i : I\!R^n \mapsto I\!R \cup \infty$, $i = 1, 2, \ldots, s$, are constraint functions such that the feasible domain $\mathcal{S}$ of (1) is convex, and it is assumed that

$$(x_1, x_2) \ne (0, 0) \quad \text{for all} \quad x \in \mathcal{S}. \tag{2}$$

Optimization problems that lead to programs of the form (1) arise in important applications. For instance, concave-convex fractional programs [8] or condition-number optimization over cones of positive definite matrices [2] can be reformulated as problems of the form (1). In the sequel, we always denote by $f_0(x) := x_2/x_1$ the objective function of (1). We stress that f_0 is not convex, but only pseudo-convex on S, and in general, the program (1) is a nonconvex nonlinear problem.

In Section 2, we formulate our assumptions on (1). In Section 3, we state our interior-point method, and in Section 4, we present a result that shows polynomial-time convergence of this algorithm.

2 Assumptions

The feasible domain $\mathcal{S}$ of (1) is assumed to be a bounded convex set in $I\!R^n$ with nonempty interior $\mathcal{S}^\circ$. The constraint functions f_i, $i = 1, 2, \ldots, s$, are required to be three times continuously differentiable on $\mathcal{S}^\circ$.

Associated with the feasible set $\mathcal{S}$ of (1) is the *logarithmic barrier function* ϕ defined by

$$\phi(x) := -\sum_{i=1}^{s} \log(-f_i(x)) \quad \text{for all} \quad x \in \mathcal{S}^\circ. \tag{3}$$

For each $x \in \mathcal{S}^\circ$ and $h \in I\!R^n$, we define a function $\psi = \psi_{x,h}$ in an open interval about $\beta = 0$ by setting

$$\psi(\beta) = \psi_{x,h}(\beta) := \phi(x + \beta h) \quad \text{for all} \quad \beta \in I\!R \quad \text{with} \quad x + \beta h \in \mathcal{S}^\circ.$$

In slight variation of the definition of Nesterov and Nemirovsky [7], we call the function ϕ *m-self-concordant* if ϕ is convex on $\mathcal{S}^\circ$ and if there is a constant $m > 0$ such that the functions $\psi = \psi_{x,h}$ satisfy the differential inequalities

$$|\psi'''(0)| \le 2\psi''(0)^{3/2} \quad \text{and} \quad |\psi'(0)| \le \sqrt{m}\sqrt{\psi''(0)}$$
$$\text{for all} \quad x \in \mathcal{S}^\circ \quad \text{and} \quad h \in I\!R^n.$$

Nesterov and Nemirovsky [7] showed that m-self-concordance of the barrier function is the crucial condition for the existence of polynomial-time interior-point methods.

In the following, we always assume that the logarithmic barrier function ϕ associated with (1) is m-self-concordant for some constant $m \ge 1$; the number m is called the *self-concordance parameter* of (1). We stress that this condition is fulfilled in many important cases; see [2] and the references given there. For instance, ϕ is m-self-concordant with self-concordance parameter $m = s$ if all constraint functions f_i in (1) are linear or convex quadratic; see [7, 5].

3 The algorithm

In a Phase I of our algorithm, we first check whether the infimum of the objective function $f_0(x)$ of (1) is $-\infty$. If this is the case, then we stop. Otherwise, we determine the quadrant in $I\!R^2$ that contains the first two components x_1 and x_2 of all optimal solutions of (1). We remark that, by (2) and the convexity of $\mathcal{S}$, the first two components of all optimal solutions lie in the same quadrant. Without loss of generality, we can assume that this is the first quadrant in $I\!R^2$, and we then replace $\mathcal{S}$ in (1) by $\mathcal{S} \cap \{ x \in I\!R^n \mid x_1 \ge 0,\ x_2 \ge 0 \}$. This, together with (2), guarantees that the feasible set $\mathcal{S}$ of (1) now satisfies

$$x_1, x_2 \ge 0 \quad \text{and} \quad (x_1, x_2) \ne (0,0) \quad \text{for all} \quad x \in \mathcal{S}. \tag{4}$$

We remark that Phase I can be carried out by solving at most three convex programming problems: (i) finding a strictly feasible point of $\mathcal{S}$; (ii) minimizing or maximizing x_1 over $\mathcal{S}$; (iii) minimizing or maximizing x_2 over $\mathcal{S}$. All three problems can be solved in polynomial time using any of the algorithms described in [7, 5].

Next we describe Phase II of our algorithm. We set

$$v(t) := [\,-\sin t \quad \cos t \quad 0 \quad \cdots \quad 0\,]^T \in I\!R^n$$

. In view of (4), we have, for any $x \in \mathcal{S}$ and $t \in [0, \pi/2]$, that $x^T v(t) \le 0$ if, and only if, $f_0(x) \le \tan t$. This shows that the fractional program (1) is equivalent to the problem

$$\begin{array}{ll} (t_{\mathrm{opt}} :=) & \text{minimize}_{t \in [0,\pi/2],\, x \in I\!R^n} \;\; t \\ & \text{subject to} \quad x \in \mathcal{S}(t) := \{\, x \in \mathcal{S} \mid x^T v(t) \le 0 \,\}. \end{array} \tag{5}$$

Clearly, $\mathcal{S}(\pi/2) = \mathcal{S}$, and $\mathcal{S}(t_{\mathrm{opt}})$ is just the set of all optimal solutions of the original problem (1). Furthermore, we have

$$\mathcal{S}^\circ(t) := \{\, x \in \mathcal{S}^\circ \mid x^T v(t) < 0 \,\} \ne \emptyset \quad \text{for all} \quad t \in (t_{\mathrm{opt}}, \pi/2].$$

Finally, for each fixed $t \in (t_{\mathrm{opt}}, \pi/2]$, we define a logarithmic barrier function $\varphi(\cdot, t)$ for $S^\circ(t)$ by setting

$$\varphi(x,t) := \phi(x) - m \log\left(-x^T v(t)\right) \quad \text{for all} \quad x \in S^\circ(t).$$

One can show [2] that the m-self-concordance of the barrier function ϕ of $\mathcal{S}$ implies that $\varphi(\cdot, t)$ is $(2m)$-self-concordant. Moreover, for each t, the function $\varphi(\cdot, t)$ is strictly convex, and we denote by $x(t)$ the unique solution of the unconstrained optimization problem

$$\underset{x \in \mathcal{S}(t)}{\text{minimize}} \;\; \varphi(x,t). \tag{6}$$

The point $x(t)$ is called the *analytic center* of $\mathcal{S}(t)$.

The formulation (5) suggests the following interior-point method for solving (1). We construct a sequence

$$\{t_k\}_{k=0}^{\infty}, \quad \text{where} \quad t_{\mathrm{opt}} < t_{k+1} < t_k \le \pi/2 \quad \text{for all} \quad k,$$

and for each k, we determine an interior point $x^{(k)}$ of $\mathcal{S}(t_k)$. If $\lim_{k\to\infty} t_k = t_{\mathrm{opt}}$, then $f_0(x^{(k)})$ converges to the optimal value of (1). The interior point $x^{(k)}$ is chosen as a crude approximation to the analytic center $x(t_k)$ of $\mathcal{S}(t_k)$ that is obtained by a single step of Newton's method (without line search) applied to (6) (with t replaced by t_k), using the previous iterate $x^{(k-1)}$ as starting guess. We denote by

$$g(x,t) := \nabla_x \varphi(x,t) \quad \text{and} \quad B(x,t) := \left(\nabla_x^2 \varphi(x,t)\right)^{-1}$$

the gradient and the inverse of the Hessian of $\varphi(\cdot, t)$, respectively. Note that $B = B(x,t)$ is a symmetric positive definite $n \times n$ matrix, and we set $\|x\|_B := \sqrt{x^T B x}$ for all $x \in \mathbb{R}^n$.

After these preliminaries, Phase II of our interior-point method can now be summarized as follows.

Algorithm 1 (Interior-Point Method for Fractional Programs with Convex Constraints.)

0) Choose $t_0 \in (0, \pi/2]$ such that $\mathcal{S}^\circ(t_0) \neq \emptyset$.
Determine a point $x^{(0)} \in \mathcal{S}^\circ(t_0)$ such that

$$\|g(x^{(0)}, t_0)\|_{B(x^{(0)},t_0)} \leq \frac{1}{16}. \tag{7}$$

For $k = 0, 1, \ldots,$ do:

1) Set

$$\Delta t_k = \frac{\left|(x^{(k)})^T v(t_k)\right|}{18m\sqrt{(x_1^{(k)})^2 + (x_2^{(k)})^2}} \quad \text{and} \quad t_{k+1} = t_k - \Delta t_k.$$

2) Set

$$x^{(k+1)} = x^{(k)} - B(x^{(k)}, t_{k+1})\, g(x^{(k)}, t_{k+1}).$$

Note that in step 0) of Algorithm 1, $t_0 = \pi/2$ is always a feasible choice. Furthermore, finding a point $x^{(0)} \in \mathcal{S}^\circ(t_0)$ with (7) requires the approximate solution of the unconstrained convex optimization problem (6) (with $t = t_0$); this can be done in polynomial time, using any of the algorithms described in [7, 5].

In [2], we establish various properties of Algorithm 1. In particular, we show that

$$t_{\text{opt}} < t_k \leq \pi/2 \quad \text{and} \quad x^{(k)} \in \mathcal{S}^\circ(t_k) \quad \text{for all} \quad k = 0, 1, \ldots. \tag{8}$$

Moreover, it turns out that the estimate (7) actually holds for all iterates, i.e., we have

$$\|g(x^{(k)}, t_k)\|_{B(x^{(k)},t_k)} \leq \frac{1}{16} \quad \text{for all} \quad k = 0, 1, \ldots. \tag{9}$$

The proofs of (8) and (9) make use of two key properties: the $(2m)$-self-concordance of $\varphi(\cdot, t)$, and a Lipschitz continuity of $g(x,t)$ and $B(x,t)$ with respect to t.

4 Polynomial convergence

In [2], we also show that Algorithm 1 converges in polynomial time. For the proof, we construct a sequence of sets $\mathcal{C}_k \in I\!R^2$, $k = 0, 1, \ldots$, such that each set $\mathcal{C}_k$ contains the first two components $x_1^{(k)}$ and $x_2^{(k)}$ of the kth iterate, and its area tends to zero as $k \to \infty$. The main results from [2] are summarized in the following theorem.

Theorem 2 *Let $x^{(k)}$ be the kth iterate generated by Algorithm 1. Let m be the self-concordance parameter of the logarithmic barrier function (3) associated with the feasible set $\mathcal{S}$ of the fractional program (1). Then:*

(*a*) *For all $k = 0, 1, \ldots$:*

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \in \mathcal{C}_k^\circ \quad \textit{and} \quad \mathcal{C}_{k+1} \subset \mathcal{C}_k;$$

$$\operatorname{area}(\mathcal{C}_k) \;\le\; \left(1 - \frac{1}{80\,m}\right)^k \left((x_1^{(0)})^2 + (x_2^{(0)})^2\right) \frac{\pi\,(1+\sqrt{2m})^4}{4}. \tag{10}$$

(*b*)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{C}_\infty := \bigcap_{k=0}^{\infty} \mathcal{C}_k \quad \textit{for all optimal solutions } x \textit{ of } (1);$$

$$\frac{x_2}{x_1} = \min_{y \in \mathcal{S}} f_0(y) \quad \textit{for all} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{C}_\infty \setminus \{0\}.$$

Clearly, the estimate (10) shows that Algorithm 1 converges in polynomial time. We remark that, by introducing an additional artificial constraint to the reformulation (5) of the fractional program (1) it is possible to devise an interior-point method similar to Algorithm 1 for which the corresponding convergence estimate (10) depends on $1/\sqrt{m}$, rather than on $1/m$; we refer the reader to [4] for details. Finally, we note that extensions of Algorithm 1 to the problem of minimizing the maximum of several fractions over convex sets are studied in [3].

References

[1] den Hertog, D.; Jarre, F.; Roos, C.; Terlaky, T.: A unifying investigation of interior-point methods for convex programming. Report Nr.

92–89, Delft University of Technology, Faculty of Technical Mathematics and Informatics, Delft 1992.

[2] Freund, R. W.; Jarre, F.: An interior-point method for convex fractional programming. AT&T Numerical Analysis Manuscript No. 93–03, Bell Laboratories, Murray Hill 1993.

[3] Freund, R. W.; Jarre, F.: An interior-point method for multi-fractional programs with convex constraints. AT&T Numerical Analysis Manuscript No. 93–07, Bell Laboratories, Murray Hill 1993.

[4] Freund, R. W.; Jarre, F.: On the complexity of fractional programs with convex constraints. In preparation.

[5] Jarre, F.: Interior-point methods for convex programming. Appl. Math. Optim. **26** (1992), 287–311.

[6] Karmarkar, N.: A new polynomial-time algorithm for linear programming. Combinatorica **4** (1984), 373–395.

[7] Nesterov, J. E.; Nemirovsky, A. S.: Self-concordant functions and polynomial-time methods in convex programming. Report, Central Economical and Mathematical Institute, USSR Acad. Sci., Moscow 1989.

[8] Schaible, S.: A survey of fractional programming. In: Schaible, S; Ziemba, W. T. (eds.): Generalized concavity in optimization and economics. Academic Press, New York 1981, pp. 417-440.