On interior-point methods for fractional programs and their convex reformulation

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Abstract

Given a self-concordant barrier function for a convex set $\mathcal{S}$, we determine a self-concordant barrier function for the conic hull $\tilde{\mathcal{S}}$ of $\mathcal{S}$. As our main result, we derive an "optimal" barrier for $\tilde{\mathcal{S}}$ based on the barrier function for $\mathcal{S}$. Important applications of this result include the conic reformulation of a convex problem, and the solution of fractional programs by interior-point methods. The problem of minimizing a convex-concave fraction over some convex set can be solved by applying an interior-point method directly to the original nonconvex problem, or by applying an interior-point method to an equivalent convex reformulation of the original problem. Our main result allows to analyze the second approach showing that the rate of convergence is of the same order in both cases.

Key words: Convex set; Conic hull; Barrier function; Interior-point method; Fractional programs; Self-concordance; Convergence rate

1. Conic hulls and fractional programming

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a given convex set. We call the set defined by

$$\tilde{\mathcal{S}} := \left\{ \begin{bmatrix} y \\ t \end{bmatrix} \in \mathbb{R}^{n+1} \;\middle|\; t > 0 \text{ and } \frac{y}{t} \in \mathcal{S} \right\} \tag{1}$$

the *conic hull* of $\mathcal{S}$.

This note is concerned with the problem of constructing an "optimal" self-concordant barrier function for the conic hull $\tilde{\mathcal{S}}$ of $\mathcal{S}$ from a given self-concordant barrier function for $\mathcal{S}$. This problem arises, for example, when fractional programs are solved by applying an interior-point method to a convex reformulation of the fractional program. In fact, our study of this problem was motivated by this application, which we now describe in more detail.

Consider fractional programs written in the standard form

$$\operatorname*{minimize}_{x\in I\!R^n} \ \frac{x_2}{x_1} \qquad \text{subject to} \quad f_i(x) \le 0, \quad i = 1, 2, \ldots, m, \tag{2}$$

where x_1 and x_2 denote the first two components of the vector x. We assume that the feasible set

$$\mathcal{S} := \{\, x \in I\!R^n \mid f_i(x) \le 0,\ i = 1, 2, \ldots, m \,\} \tag{3}$$

of (2) is a convex set with nonempty interior,

$$\mathcal{S}^\circ \ne \emptyset, \tag{4}$$

and that

$$x_1 \ \ge\ 0 \quad \text{and} \quad (x_1, x_2) \ne (0, 0) \quad \text{for all} \quad x \in \mathcal{S}. \tag{5}$$

Note that, while $\mathcal{S}$ is assumed to be convex, the constraint functions f_i in (3) themselves are not required to be convex. In [5], it was pointed out that (2)–(5) is a standard form for various important classes of fractional programs such as convex-concave fractional programs. For applications of fractional programs, we refer the reader to [13], which also contains a comprehensive bibliography of fractional programming.

As Schaible [10, 11, 12] pointed out in an effort to establish a duality theory, convex-concave fractional programs can be reformulated as convex problems, by means of a generalization of the transformation used by Charnes and Cooper [2, 1] for linear fractional programs. For fractional programs in the standard form (2), the generalized Charnes-Cooper transformation introduced in [10] is especially simple, and it is given by

$$t = \frac{1}{x_1} \quad \text{and} \quad y = tx. \tag{6}$$

With (6) and using the assumptions (4) and (5), one readily verifies that the problem (2) is equivalent to either one of the following programs

$$\operatorname*{minimize}_{t\in I\!R,\, y\in I\!R^n} \ y_2 \qquad \text{subject to} \quad t > 0, \ \ y_1 = 1, \ \ \text{and} \ \ f_i\left(\frac{y}{t}\right) \le 0, \ \ i = 1, 2, \ldots, m, \tag{7}$$

and

$$\operatorname*{minimize}_{t\in I\!R,\, y\in I\!R^n} \ y_2 \qquad \text{subject to} \quad t > 0, \ \ y_1 = 1, \ \ \text{and} \ \ t f_i\left(\frac{y}{t}\right) \le 0, \ \ i = 1, 2, \ldots, m. \tag{8}$$

Here, equivalence means that the optimal values of (2), (7), and (8) are identical. We remark that, in view of the strict inequality $t > 0$ in (7) and (8), it is possible that the optimal value in (2) is attained for a point in $\mathcal{S}$, while the optimal value in (7) and (8) is not attained. In fact, using (4) and (5), one immediately verifies that this case occurs if, and only if, there exists a point $x \in \mathcal{S}$ with $x_1 = 0$ and $x_2 < 0$; note that the optimal value is $-\infty$ in this case.

In view of (1) and (3), the *restricted* conic hull

$$\overline{\mathcal{S}} := \tilde{\mathcal{S}} \cap \left\{ \begin{bmatrix} y \\ t \end{bmatrix} \in I\!R^{n+1} \;\middle|\; y_1 = 1 \right\} \tag{9}$$

is just the feasible set of (7), and obviously, also of (8). In [10, 11], it was shown that $\tilde{\mathcal{S}}$ and thus $\overline{\mathcal{S}}$ are convex sets, and therefore, both reformulations (7) and (8) of the original problem (2) are convex programs. If the constraint functions f_i of (2) are all convex, then the reformulation (8) has the advantage over (7) that the constraint functions of (8) are also convex. The constraint functions of (7), however, need not be convex; for example, if $f_i(x) = -x$: $I\!R_+ \to I\!R$, then the function $f_i(y/t) = -y/t$: $I\!R_+^2 \to I\!R$ is not convex.

In the following lemma, we summarize these results on the reformulations (7) and (8).

Lemma 1. (Schaible [10, 11])

a) *The conic hull $\tilde{\mathcal{S}}$ of a convex set $\mathcal{S} \subseteq I\!R^n$ and the set $\overline{\mathcal{S}}$ defined in* (9) *are convex.*

b) *If the constraint function f_i for problem* (2) *is convex on the feasible set $\mathcal{S}$ defined in* (3), *then the constraint function $tf_i(y/t)$ for problem* (8) *is convex on $\tilde{\mathcal{S}}$.*

Proof. The proof of part a) is straightforward. For the proof of part b), see, e.g., [4, Pages 160–161]. □

The remainder of the paper is organized as follows. In Section 2, we briefly review the concept of self-concordant barrier functions. In Section 3, we state and prove our main result. Finally, in Section 4, we discuss some consequences of this result, and we make some other concluding remarks.

Throughout this paper, $\mathcal{S} \subseteq I\!R^n$ is assumed to be a convex set with nonempty interior (with respect to the Euclidean topology of $I\!R^n$) $\mathcal{S}^\circ$, and $\tilde{\mathcal{S}}$ denotes the conic hull of $\mathcal{S}$ defined in (1).

2. Self-concordant barrier functions

Most interior-point methods are based on some barrier function F for the convex set $\mathcal{S}$. The function F is a *barrier function* for $\mathcal{S}$ if it is finite on $\mathcal{S}^\circ$ and it tends to ∞ as the argument x of F approaches the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. Additional properties of F are crucial for the performance of the interior-point method. Two essential properties that allow the construction of *polynomial-time* methods, and that are also of importance for numerical implementations are due to Nesterov and Nemirovskii [9]. The first property is called *self-concordance*, and it can be stated as follows.

Definition 2. *Let $F : \mathcal{S}^\circ \to I\!R$ be a convex function that is three times continuously differentiable. The function F is called self-concordant if, for any point $x \in \mathcal{S}^\circ$ and any direction h, the derivatives of the function ψ defined by $\psi(\tau) = F(x + \tau h)$ satisfy*

$$|\psi'''(0)| \le 2(\psi''(0))^{3/2}. \tag{10}$$

Note that, since $x \in \mathcal{S}^\circ$ and $\mathcal{S}^\circ$ is open, the domain of ψ contains some open interval about $\tau = 0$, and thus the quantities in (10) are well defined.

The condition (10) is equivalent to a relative Lipschitz condition of the Hessian of F, and it guarantees that Newton's method for minimizing F converges quadratically in the ellipsoid

$$\mathcal{E} = \{\, x \mid (x - \bar{x})^T \nabla^2 F(\bar{x})(x - \bar{x}) \le 1/4 \,\},$$

assuming that a unique minimizer $\bar{x}$ of F exists; see, e.g., [8].

The second condition that is crucial for the design of polynomial-time algorithms will be referred to as *self-limitation with parameter* μ, and it can be stated as follows.

Definition 3. *Let $F : \mathcal{S}^\circ \to I\!R$ be a convex function that is twice continuously differentiable. The function F is called self-limiting with parameter μ if, for any point $x \in \mathcal{S}^\circ$ and any direction h, the derivatives of the function ψ defined by $\psi(\tau) = F(x + \tau h)$ satisfy*

$$(\psi'(0))^2 \leq \mu \psi''(0). \tag{11}$$

In [9], barrier functions that satisfy both conditions (10) and (11) are called μ*-self-concordant* barrier functions. We remark that there is no μ-self-concordant barrier function with $\mu < 1$—except the constant function on $I\!R^n$. While the parameter μ cannot be less than one, typically it is of "reasonable size". For example, for linear or quadratic inequality constraints, one can show that $\mu = m$ suffices, where m is the number of constraints. In particular, if all the functions f_i in the definition of $\mathcal{S}$ in (3) are linear or quadratic, then

$$F(x) = -\sum_{i=1}^{m} \log(-f_i(x))$$

provides such a self-concordant barrier function.

Combining conditions (10) and (11) allows the construction of a second concentric ellipsoid containing $\mathcal{S}$, namely

$$\mathcal{S} \subset \tilde{\mathcal{E}} = \{ x \mid (x - \bar{x})^T \nabla^2 F(\bar{x})(x - \bar{x}) \leq (\mu + 2\sqrt{\mu})^2 \};$$

see [7]. This result implies that the domain of quadratic convergence of Newton's method, which contains $\mathcal{E}$, is not too small compared to the domain $\mathcal{S}^\circ$ of F.

Moreover, the above results—when applied to a barrier method—allow the proof of a polynomial rate of convergence, namely that a certain measure for the distance to optimality (usually the gap in the objective function to its optimal value, if the iterates are feasible) is reduced by a factor 2 within at most $\mathcal{O}(\sqrt{\mu})$ iterations of the barrier method. To date, the concept of self-concordance is the most general framework that allows the construction of polynomial-time methods in the above sense. In the next section, we will investigate the transformation of Section 1 with respect to self-concordance. We will show that not only convexity, but also the availability of a self-concordant barrier function, is preserved when the original problem (2) is reformulated as (7) or (8). This observation is interesting for the design of algorithms for solving (2), and for solving the conic reformulation of a convex problem; see, e.g., [9].

3. Main result

In this section, we present our main result, namely how to construct an "optimal" self-concordant barrier function F^+ for the conic hull $\tilde{\mathcal{S}}$ (defined in (1)) from a given μ-self-concordant barrier function F for the convex set $\mathcal{S}$. Since linear equality constraints like $y_1 = 1$ do not affect the self-concordance properties (see [9, Proposition 2.1.1]), the barrier function F^+ for $\tilde{\mathcal{S}}$ can also be used for an interior-point method for solving the convex problems (7) and (8) over the feasible set $\overline{\mathcal{S}}$ defined in (9).

The barrier function F^+ is constructed as a positive linear combination

$$F^+(y,t) = \vartheta F\left(\frac{y}{t}\right) - \vartheta\alpha\mu\log t, \quad \text{where} \quad \vartheta,\ \alpha \in I\!R, \quad \vartheta,\ \alpha > 0, \tag{12}$$

of F and the 1-self-concordant barrier function $-\log t$ for the set $\{\, t \mid t \geq 0 \,\}$. In (12), the parameters ϑ and α are still arbitrary. It is known (see, e.g., [9, Section 5.1.1]) that F^+ is a self-limiting barrier function with parameter $\vartheta\alpha\mu$ if $\alpha > 1$. In Theorem 4 below, we determine a good choice for the parameters ϑ and α. In some sense this choice is best possible; it gives the lowest value of the self-concordance parameter of F^+ using only certain inequalities on the derivatives of F in certain directions based on the knowledge of its self-concordance parameter μ.

Barrier functions of the form (12) were first considered by Nesterov and Nemirovskii; see [9, Section 5.1.1]. However, there it is merely shown that, for the parameters $\vartheta = 400$ and $\alpha = 2$, the function F^+ is self-concordant with self-concordance parameter 800μ, and no attempt is made to find the minimal self-concordance parameter.

When the barrier function (12) is used for a numerical implementation, it may be crucial to use the best-possible (or a very good) value for the self-concordance parameter, since this parameter is one of the factors that determine the efficiency of the implementation.

We point out, however, that our function F^+ is, in general, not the best-possible barrier for $\tilde{\mathcal{S}}$ (even if F happened to be best-possible for $\mathcal{S}$); it is the best-possible one that can be obtained as a linear combination of F and $-\log t$ using only certain inequalities based on the self-concordance parameter μ of F. In fact, there can be better linear combinations of F and $-\log t$. For example, consider the feasible set $\mathcal{S} = I\!R^n_+$ with the standard self-concordant barrier function

$$F(x) = -\sum_{i=1}^{n} \log(x_i).$$

For this example, $\mu = n$, and by results of [9], the barrier function (12) with parameters $\vartheta = 1$ and $\alpha = 1 + 1/n$ is optimal for the conic hull $\tilde{\mathcal{S}} = I\!R^{n+1}_+$ of $\mathcal{S}$.

Theorem 4. *Let $\mathcal{S} \in I\!R^n$ be a convex set with nonempty interior $\mathcal{S}^\circ$. Assume that $\mathcal{S}$ has a self-concordant barrier function $F : \mathcal{S}^\circ \to I\!R$ with parameter μ (≥ 1). Then, the conic hull $\tilde{\mathcal{S}}$ of $\mathcal{S}$ has a self-concordant barrier function $F^+(y,t)$ of the form* (12)*. For the parameter values*

$$\alpha = 4 + \frac{3}{\sqrt{\mu}} \quad \text{and} \quad \vartheta = \frac{1}{\alpha\mu}\left(1 + 9\frac{\alpha - 1}{\left(\alpha\sqrt{\mu}(\sqrt{\alpha} - 2) - 3\right)^2}\right), \tag{13}$$

the barrier function (12) *is convex and self-concordant with parameter $\mu\gamma(\mu)$, where the function γ is defined by*

$$\gamma(\mu) = \frac{1}{\mu} + \frac{1}{9}\left(\sqrt{\alpha(\mu)} + 2\right)^4 \frac{\sqrt{\alpha(\mu)} - 1}{\sqrt{\alpha(\mu)} + 1}, \quad \alpha(\mu) = 4 + \frac{3}{\sqrt{\mu}}. \tag{14}$$

Furthermore, the function γ is strictly monotonically decreasing for $\mu \geq 1$, with

$$\max_{\mu \geq 1} \gamma(\mu) = \gamma(1) = \frac{343 + 119\sqrt{7}}{27} \approx 24.365 \tag{15}$$

and

$$\inf_{\mu\geq 1} \gamma(\mu) = \lim_{\mu\to\infty} \gamma(\mu) = \frac{256}{27} = 9.\overline{481}\ . \tag{16}$$

Moreover, the parameter values (13) *for* ϑ *and* α *are best possible* (*for minimizing the self-concordance parameter of* (12)), *if all that is used are the inequalities* (19) *below that follow from the self-concordance of* F.

Proof. We start by following the proof and notation of [9, Proposition 5.1.4, Page 150], and we derive explicit expressions (equations (17) and (18) below) for the directional derivatives of the function $F^+(y,t)$ defined in (12). Here, we treat ϑ and α as arbitrary positive constants whose values have to be determined such that F^+ is convex and self-concordant, and such that the self-concordance parameter is minimal.

Let $[\,y^T \quad t\,]^T \in \tilde{\mathcal{S}}$ be arbitrary, but fixed, and let $[\,h^T \quad -s\,]^T$ be an arbitrary, but fixed direction. We set $\psi(\tau) = F^+(y+\tau h, t-\tau s)$ as in (10), (11), and with

$$\frac{y+\tau h}{t-\tau s} = \frac{y}{t} + r\frac{\sigma y + h}{t}, \quad \text{where} \quad \sigma = \frac{s}{t}, \quad r = \frac{\tau}{1-\tau\sigma},$$

we obtain

$$d_2 := \psi''(0) = \vartheta(\alpha\mu\sigma^2 + 2\sigma\pi_1 + \pi_2^2) \tag{17}$$

and

$$d_3 := \psi'''(0) = \vartheta(2\alpha\mu\sigma^3 + 6\sigma^2\pi_1 + 6\sigma\pi_2^2 + \pi_3). \tag{18}$$

Here,

$$\pi_1 = \phi'(0), \quad \pi_2 = (\phi''(0))^{1/2} > 0, \quad \text{and} \quad \pi_3 = \phi'''(0),$$

with ϕ the function defined by

$$\phi(r) = F\left(\frac{y}{t} + r\frac{\sigma y + h}{t}\right).$$

Since F is a μ-self-concordant barrier function, we have the two inequalities

$$\mu\pi_2^2 \geq \pi_1^2 \quad \text{and} \quad 2\pi_2^3 \geq |\pi_3|. \tag{19}$$

We will show that the parameters α and ϑ in (13) provide the best-possible values when all that we can use about F are the inequalities (19), by solving an optimization problem where these provide the constraints.

First, we prove the convexity of F^+. To this end, we need to show that $\psi''(0) \geq 0$. From (17) and (19), we get

$$\psi''(0) = d_2 \geq \vartheta(\alpha\mu\sigma^2 - 2|\sigma|\sqrt{\mu}\pi_2 + \pi_2^2) = \vartheta((\pi_2 - \sqrt{\mu}|\sigma|)^2 + (\alpha-1)\mu\sigma^2) \geq 0 \quad \text{if} \quad \alpha \geq 1,$$

and therefore, F^+ is convex if $\alpha \geq 1$. However, as the example $F(y) := -\sum_{i=1}^n \log y_i$ with $\mu = n$ shows, the value $\alpha = 1$ is not sufficient in general to guarantee the barrier property of F^+. In the sequel, we therefore assume that $\alpha > 1$.

Next, we note that, for $\alpha > 1$, the barrier property of F^+, i.e., $\lim_{(y,t)\to\partial\tilde{\mathcal{S}}} F^+(y,t) = \infty$ is shown in [9]. Furthermore, in [9], it is proved that the self-limitation property (11) holds with parameter $\vartheta\alpha\mu$, i.e.,

$$(\psi'(0))^2 \leq \vartheta\alpha\mu\ d_2. \tag{20}$$

Therefore, it only remains to show that ψ satisfies the self-concordance property (10), i.e.,

$$4d_2^3 \geq d_3^2. \tag{21}$$

As the self-concordance parameter in (20) is $\vartheta\alpha\mu$ where the value of μ is given, our goal is to find the values of the parameters $\vartheta > 0$ and $\alpha > 1$ that minimize $\vartheta\alpha$ subject to (19) and (21). To this end, we first show that one can set, without loss of generality, $\pi_3 = +2\pi_2^3$ when proving (21).

For $\sigma = 0$, (21) reduces to $\vartheta \geq 1$ because of (17), (18), and (19). Hence we assume from now on that $\sigma \neq 0$ (and $\vartheta \geq 1$). By temporarily writing d_2 and d_3 in terms of the artificial variables $v_1 := \sigma\pi_1$ and $v_3 := \pi_3/\sigma$, we obtain from (17) and (18) that

$$d_2 = \vartheta(\alpha\mu\sigma^2 + 2v_1 + \pi_2^2) \quad \text{and} \quad d_3 = \vartheta\sigma(2\alpha\mu\sigma^2 + 6v_1 + 6\pi_2^2 + v_3),$$

and thus the two quantities d_2^3 and d_3^2 appearing in (21) only involve even powers of σ. Hence, we can assume that, without loss of generality, σ is restricted to be positive, $\sigma > 0$. (Note that π_1 and π_3 are unrestricted in sign, so that for any valid v_1 and v_3 and $\sigma > 0$, we can find appropriate values for π_1 and π_3.)

From $\mu\pi_2^2 \geq \pi_1^2$, $\alpha > 1$ and $\sigma > 0$, it follows that the three terms in (18),

$$2\alpha\mu\sigma^3 + 6\sigma^2\pi_1 + 6\sigma\pi_2^2 \geq 2\alpha\mu\sigma^3 - 6\sigma^2\sqrt{\mu}\pi_2 + 6\sigma\pi_2^2 \geq 2\sigma(\sqrt{\mu}\sigma - \frac{3}{2}\pi_2)^2 \geq 0,$$

are nonnegative, and hence $|d_3|$ is maximized if the fourth term π_3 is maximal with respect to (19), $\pi_3 = +2\pi_2^3$. Note that d_2 does not depend on π_3, and thus we set $\pi_3 = +2\pi_2^3$ from now on.

For our further analysis, we use the notations

$$\kappa = \alpha\mu > 1, \quad \pi = \frac{\pi_1}{\pi_2}, \quad \text{and} \quad \nu = \frac{\sigma}{\pi_2} > 0,$$

and we rewrite d_2 and d_3 as

$$d_2 = \vartheta\pi_2^2\left(\kappa\nu^2 + 2\pi\nu + 1\right) \quad \text{and} \quad d_3 = 2\vartheta\pi_2^3\left(\kappa\nu^3 + 3\pi\nu^2 + 3\nu + 1\right).$$

Furthermore, we set

$$p(\nu) = \kappa\nu^3 + 3\pi\nu^2 + 3\nu + 1,$$

and then (21) can be reformulated as

$$4 \geq \frac{d_3^2}{d_2^3} \quad \Longleftrightarrow \quad \frac{\vartheta}{27} \geq \frac{(p(\nu))^2}{(p'(\nu))^3}. \tag{22}$$

Note that, as $\nu \to \infty$, the condition (22) reduces to $\vartheta\kappa \geq 1$, which is clearly satisfied. Likewise, for $\nu \to 0$, it reduces to $\vartheta \geq 1$. Thus, it is sufficient to concentrate on the finite values of $\nu > 0$ that maximize the right-hand side of (22). By differentiating with respect to ν, we find that the $\nu > 0$ that maximizes the right-hand side of (22) satisfies

$$2(p'(\nu))^2 - 3p(\nu)p''(\nu) = 0, \tag{23}$$

which is equivalent to

$$(\pi^2 - \kappa)\nu^2 + (\pi - \kappa)\nu + 1 - \pi = 0, \qquad \nu > 0. \tag{24}$$

The relation (23) also implies that

$$\frac{(p(\nu))^2}{(p'(\nu))^3} = \frac{2p(\nu)}{3p'(\nu)p''(\nu)} \quad \text{and} \quad (22) \quad \Longleftrightarrow \quad \frac{\vartheta}{18} \geq \frac{p(\nu)}{p'(\nu)p''(\nu)}.$$

The minimal factor $\vartheta\alpha$ for the self-concordance parameter $\vartheta\alpha\mu$ of F^+ is thus given by

$$\begin{aligned}
\min\ \vartheta\alpha &= \min_{\kappa>\mu,\ \vartheta\geq 1} \left\{ \max_{\pi,\nu>0} \left\{ \vartheta\,\frac{\kappa}{\mu} \,\middle|\, -\sqrt{\mu} \leq \pi \leq \sqrt{\mu},\ \frac{\vartheta}{18} \geq \frac{p(\nu)}{p'(\nu)p''(\nu)},\ \nu \text{ satisfies } (24) \right\} \right\} \\
&= \min_{\kappa>\mu} \left\{ \max_{\pi,\nu>0} \left\{ \frac{18p(\nu)}{p'(\nu)p''(\nu)} \frac{\kappa}{\mu} \,\middle|\, -\sqrt{\mu} \leq \pi \leq \sqrt{\mu},\ \nu \text{ satisfies } (24) \right\} \right\} \\
&= \frac{1}{\mu} \min_{\kappa>\mu} \left\{ \max_{\pi,\nu>0} \left\{ 1 + \frac{\kappa-\pi^2}{(\kappa\nu+\pi)^2} \,\middle|\, -\sqrt{\mu} \leq \pi \leq \sqrt{\mu},\ \nu \text{ satisfies } (24) \right\} \right\} \qquad (25) \\
&=: \gamma(\mu). \qquad (26)
\end{aligned}$$

It remains to determine the optimal value of (25), i.e., the function $\gamma(\mu)$.

The optimal value of (25) is determined in two steps. As a first step, we show that

$$\max_{\pi,\nu>0} \left\{ 1 + \frac{\kappa-\pi^2}{(\kappa\nu+\pi)^2} \,\middle|\, -\sqrt{\mu} \leq \pi \leq \sqrt{\mu},\ \nu \text{ satisfies } (24) \right\} \qquad (27)$$

is attained for $\pi = -\sqrt{\mu}$ independently of $\kappa > \mu$. Since (24) does not have a positive solution ν for $\pi \geq 1$, we conclude that $\pi < 1$. Inserting $\nu = 0$ into the left-hand side of (24) yields the value $1 - \pi > 0$, and also $\nu = -\pi/\kappa$ yields a positive value, namely $1 - (\pi^2/\kappa)^2$ where $\pi^2 \leq \mu < \kappa$. Since the leading coefficient $\pi^2 - \kappa < 0$ in (24) is negative, we conclude that the only positive solution ν of (24) is greater than $-\pi/\kappa$, and thus $\kappa\nu + \pi > 0$. Next, we prove that the maximum of the objective function in (27) is attained at $\pi = -\sqrt{\mu}$, by showing that its derivative with respect to π is negative. Note that, in (27), $\nu = \nu(\pi) > 0$ is a function of π defined by (24), and κ is a constant parameter. For the derivative of the objective function in (27), we obtain

$$\frac{d}{d\pi}\left(1 + \frac{\kappa-\pi^2}{(\kappa\nu+\pi)^2}\right) = -\frac{2\kappa}{(\kappa\nu+\pi)^3}A, \quad \text{where} \quad A = (\kappa-\pi^2)\frac{d\nu}{d\pi} + \pi\nu + 1. \qquad (28)$$

Since $\kappa\nu + \pi > 0$, it suffices to show that $A \geq 0$. Differentiating (24) with respect to π yields

$$\frac{d\nu}{d\pi} = \frac{2\pi\nu^2 + \nu - 1}{(\kappa-\pi^2)2\nu + \kappa - \pi}.$$

By inserting this into the definition of A in (28) and then replacing the terms of the form $(\kappa-\pi^2)\nu^2$ with $(\pi-\kappa)\nu + 1 - \pi$ (from (24)), we obtain that

$$\frac{A}{(\kappa-\pi^2)2\nu + \kappa - \pi} = 3(1-\pi)(\kappa\nu+\pi) \geq 0. \qquad (29)$$

Here, the last inequality holds true because $\pi < 1$ and $\kappa\nu + \pi > 0$. Since $\nu > 0$, $\kappa > \mu \geq \pi^2$, and $\kappa > 1 \geq \pi$, the inequality (29) implies $A \geq 0$.

Having shown that (27) is maximized by $\pi = -\sqrt{\mu}$, we start the second step by concluding that, by (26), (25), and (24), we have

$$\gamma(\mu) = \frac{1}{\mu} \min_{\kappa > \mu} \left\{ 1 + \frac{\kappa - \mu}{(\kappa\nu - \sqrt{\mu})^2} \right\}, \tag{30}$$

where ν is the positive solution of

$$(\kappa - \mu)\nu^2 + (\sqrt{\mu} + \kappa)\nu - 1 - \sqrt{\mu} = 0. \tag{31}$$

It will be convenient to use the notations

$$\lambda = \nu\sqrt{\mu} \quad \text{and} \quad c = \sqrt{\mu}, \tag{32}$$

where λ is a function of ν that in turn is a function of κ or of α, and to set

$$\beta(\alpha) = \beta_\lambda(\alpha) = 1 + \frac{\alpha - 1}{(\alpha\lambda - 1)^2}. \tag{33}$$

By (32) and (31), $\lambda = \lambda(\alpha)$ is just the positive solution of

$$(\alpha - 1)\lambda^2 + (\alpha c + 1)\lambda - (1 + c) = 0. \tag{34}$$

With (32) and (33), we can rewrite (30) as follows:

$$\gamma(\mu) = \frac{1}{\mu} \min_{\alpha > 1} \{ \beta(\alpha) \mid \lambda \text{ satisfies (34)} \}. \tag{35}$$

It remains to determine

$$\min_{\alpha > 1} \beta(\alpha), \tag{36}$$

and to this end, we need to compute

$$\beta'(\alpha) = \frac{d\beta(\alpha)}{d\alpha}.$$

(In the following $'$ always denotes differentiation with respect to α.)

For the computation of $\beta'(\alpha)$, we will also need $\lambda'(\alpha)$. By differentiating (34) with respect to α and solving for $\lambda'(\alpha)$, it follows that

$$\lambda' = \lambda'(\alpha) = \frac{-\lambda(\lambda + c)}{2\lambda(\alpha - 1) + \alpha c + 1}. \tag{37}$$

Now, we differentiate $\beta(\alpha)$ defined in (33). This gives

$$\beta'(\alpha) = \frac{1}{(\alpha\lambda - 1)^2} - 2\frac{\alpha - 1}{(\alpha\lambda - 1)^3}(\lambda + \alpha\lambda'). \tag{38}$$

Inserting λ' from (37) into (38) and multiplying through by

$$(\alpha\lambda - 1)^3(2\lambda(\alpha - 1) + \alpha c + 1),$$

we deduce from (38) that

$$
\begin{aligned}
&(\alpha\lambda-1)^3(2\lambda(\alpha-1)+\alpha c+1)\beta'(\alpha)\\
&\quad=(2\lambda(\alpha-1)+\alpha c+1)(\lambda(2-\alpha)-2\alpha(\alpha-1)\lambda'-1)\\
&\quad=4\lambda^2(\alpha-1)+\lambda\left((2-\alpha)(\alpha c+1)+2(\alpha-1)(\alpha c-1)\right)-1-\alpha c. \qquad (39)
\end{aligned}
$$

Next, we note that, by (34),

$$\lambda^2(\alpha-1)=-\lambda(\alpha c+1)+c+1. \qquad (40)$$

Using (40) to eliminate the term $\lambda^2(\alpha-1)$ in (39), we obtain

$$
\begin{aligned}
&(\alpha\lambda-1)^3(2\lambda(\alpha-1)+\alpha c+1)\beta'(\alpha)\\
&\quad=\lambda\alpha\left(c(\alpha-4)-3\right)+4(c+1)-1-\alpha c \qquad (41)\\
&\quad=(\lambda\alpha-1)\left(c(\alpha-4)-3\right).
\end{aligned}
$$

Our earlier observation $\nu>-\pi/\kappa=\sqrt{\mu}/\kappa$ implies that $\lambda>1/\alpha$. Hence from (41) it follows that

$$\beta'(\alpha)=0 \quad\Longleftrightarrow\quad \alpha-4=\frac{3}{c}.$$

Recall from (32) that $c=\sqrt{\mu}$, and therefore,

$$\alpha=4+\frac{3}{\sqrt{\mu}} \qquad (42)$$

is the value of α for which the minimum in (36) is attained. (From (41), $\beta'(\alpha)$ is negative for smaller values of α and positive for larger values, so this is indeed a minimum.) We now insert the value (42) of α into (34) and obtain

$$\frac{1}{\sqrt{\mu}}(3\sqrt{\mu}+3)\lambda^2+(4\sqrt{\mu}+4)\lambda-(\sqrt{\mu}+1)=0,$$

which (after eliminating $(1+\sqrt{\mu})$) results in the positive root,

$$\lambda=\left(\sqrt{4\mu+3\sqrt{\mu}}-2\sqrt{\mu}\right)/3=\sqrt{\mu}(\sqrt{\alpha}-2)/3. \qquad (43)$$

Given α and λ as above, we next compute the corresponding optimal value of ϑ. By (26) and (35), we have

$$\vartheta=\frac{\gamma(\mu)}{\alpha}=\frac{\beta(\alpha)}{\alpha\mu}=\frac{1}{\alpha\mu}\left(1+\frac{\alpha-1}{(\alpha\lambda-1)^2}\right). \qquad (44)$$

Inserting λ from (43) into (44) results in the formula for ϑ in (13). Straightforward substitutions result in

$$\gamma(\mu)=\frac{1}{\mu}+\frac{1}{9}\left(\sqrt{\alpha}+2\right)^4\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1} \qquad (45)$$

$$=\frac{1}{9}\left(\sqrt{\alpha}+2\right)^2\left(\left(\sqrt{\alpha}-2\right)^2+\left(\sqrt{\alpha}+2\right)^2\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1}\right). \qquad (46)$$

The formula (45) is just the one stated in (14). Since the functions

$$\left(\sqrt{\alpha}+2\right)^2, \quad \left(\sqrt{\alpha}-2\right)^2, \quad \text{and} \quad \frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1}$$

are all strictly monotonically increasing as long as $\alpha \geq 4$, the expression (46) (considered as a function of α) is also strictly monotonically increasing for $\alpha \geq 4$. Recall that $\mu \geq 1$, and note that, by (42), the function $\alpha = \alpha(\mu)$ is strictly monotonically decreasing. Therefore, the function $\gamma(\mu)$ is strictly monotonically decreasing for $\mu \geq 1$. Since $\alpha(1) = 7$, we obtain (15) and (16) from (14). Observe that the quantities $\alpha, \gamma, \vartheta$ derived above lie in the intervals as required; for example, $\alpha > 1$. This completes the proof of Theorem 4. ▯

4. Discussion

Today, the theory of interior-point methods for the solution of convex programming problems is very well developed; see [9, 3] and the references given there. Fractional problems of the type (2)–(5), however, are nonconvex, since the objective function x_2/x_1 is not convex (but pseudo-convex). In [5], Freund and Jarre developed an interior-point method for the class of nonconvex problems (2)–(5). They showed that a short-step version of their algorithm converges at a polynomial rate, and they discussed a practical implementation of the proposed method.

On the other hand, we have seen in Section 1 that fractional programs with convex constraints can be reformulated as convex programs, and in Section 3 that this reformulation also allows the construction of a best-possible self-concordant barrier from a given self-concordant barrier for the feasible set of the original problem. Consequently, problems of the form (2)-(5) can also be solved by applying an interior-point method for convex problems to the convex reformulation of the original problem. Since the self-concordance parameter corresponding to the convex reformulation is just a constant multiple (which, by (15), is at most 24.364...) of the self-concordance parameter μ of the original feasible set, there is an interior-point method that, when applied to the reformulation, needs at most $\mathcal{O}(\sqrt{\mu})$ iterations to reduce the optimality gap by a factor of 2; see, e.g., [9]. This rate of convergence is of the same order as the rate proved in [5].

We note however, that if all constraint functions f_i are of the form

$$f_i(x) = g_i(h_i(x)),$$

with h_i affine and g_i such that the set $\{\, y \in I\!R^n \mid g_i(y) \leq 0 \,\}$ is a cone, then there is a simpler reformulation of the fractional program to an equivalent convex program. Observe that, by the above condition on g_i, we have

$$\begin{aligned} & \operatorname*{minimize}_{x \in I\!R^n} \frac{x_2}{x_1} \qquad \text{subject to} \quad g_i(h_i(x)) \leq 0,\ i = 1, 2, \ldots, m, \\ \iff \quad & \operatorname*{minimize}_{x \in I\!R^n} \frac{x_2}{x_1} \qquad \text{subject to} \quad g_i\left(\frac{1}{x_1} h_i(x)\right) \leq 0,\ i = 1, 2, \ldots, m. \end{aligned} \tag{47}$$

Setting $y_1 = 1/x_1$, $y_i = x_i/x_1$ for $i \geq 2$, we see that (47) is also equivalent to

$$\operatorname*{minimize}_{y \in I\!R^n} \ y_2 \qquad \text{subject to} \quad g_i(\tilde{h}_i(y)) \leq 0,\ i = 1, 2, \ldots, m,$$

where $\tilde{h}$ is again affine in y and defined by the identity

$$\tilde{h}(y) = \tilde{h}(\frac{1}{x_1}, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}) = \frac{1}{x_1} h(x_1, x_2, \dots, x_n).$$

Finally, we note that the interior-point method in [5] for solving problem (2) in its original nonconvex formulation was extended in [6] to the case of minimizing the maximum of several fractions subject to convex constraints. We stress that, for such multi-fractional programs, a convex reformulation of the form (7) or (8) is no longer possible.

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