The Block-QMR Method for the Solution of Multiple Radiation and Scattering Problems in Structural Acoustics [1]

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ABSTRACT

Finite-element discretizations of time-harmonic acoustic wave problems in underwater structural acoustics result in large sparse systems of linear equations with complex symmetric coefficient matrices. Often, these matrix problems need to be solved repeatedly for different right-hand sides, but with the same matrix. We discuss the use of the block-QMR method for the iterative solution of these multiple linear systems.

INTRODUCTION

We are concerned with the solution of several systems of linear equations,

$$\mathbf{A}\,\mathbf{x}^{(j)} = \mathbf{b}^{(j)}, \quad j = 1, 2, \ldots, m, \tag{1}$$

which involve the same matrix $\mathbf{A}$, but different right-hand sides. Such problems are encountered in many applications. Here, we consider finite-element discretizations of time-harmonic wave problems in underwater structural acoustics, which give rise to linear systems (1) with a complex symmetric matrix $\mathbf{A}$. In these problems, multiple right-hand sides occur naturally when several acoustic sources need to be considered individually (radiation problems) and/or when multiple angles of incidence of an incoming plane wave need to be considered (scattering problems).

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ACOUSTIC RADIATION AND SCATTERING PROBLEMS

In this section, we show how multiple systems (1) arise from finite-element discretizations of two-dimensional acoustic radiation and scattering problems.
Consider an elastic structure Ω_s submerged in an infinite fluid medium Ω_∞ in two dimensions. Let the fluid-structure interface, where the acoustic pressure loads the structure, be denoted by Γ_i. Assuming harmonic time dependence $e^{-\mathrm{i}\omega t}$, where $\mathrm{i} = \sqrt{-1}$ and ω is the angular frequency, the structural displacement vector $\mathbf{u} \in \mathbb{C}^{n_{\mathrm{dof}}}$ and the scalar pressure p in the fluid satisfy:

$$\mathcal{L}_s \mathbf{u} = \mathbf{f}_s \qquad \text{in } \Omega_s, \tag{2}$$

$$-\nabla^2 p - k_0^2 p = 0 \qquad \text{in } \Omega_\infty, \tag{3}$$

$$\nabla p \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma_i, \tag{4}$$

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial p}{\partial r} - \mathrm{i} k_0 p \right) = 0. \tag{5}$$

Furthermore, $\mathbf{u}$ and p are coupled via conditions of compatibility and continuity across Γ_i:

$$\mathcal{B}\,\mathbf{u} = -p\,\mathbf{n} \qquad \text{on } \Gamma_i, \tag{6}$$

$$\nabla p \cdot \mathbf{n} = \rho_0 \omega^2 \mathbf{u} \cdot \mathbf{n} \qquad \text{on } \Gamma_i. \tag{7}$$

In (2)–(7), $\mathbf{f}_s$ is an externally-applied load on the structure, k_0 is the acoustic wave number, ρ_0 is the fluid density, and $\mathbf{n}$ is the unit outward (from Ω_∞) normal vector at Γ_i. The differential operator $\mathcal{L}_s$ in (2) represents the governing differential equations of the structure, which may be modeled using linear elasticity or under suitable assumptions of shell, plate, or beam theory, and the operator $\mathcal{B}$ is used to enforce natural boundary conditions on $\mathbf{u}$. (Accordingly, n_{dof} denotes the number of degrees of freedom of the structural model.)
In order to obtain a finite computational problem, the unbounded fluid domain exterior to the fluid-structure interface needs to be truncated. This is achieved by introducing an artificial circular boundary Γ_∞ that completely encloses the structure and defines a finite fluid domain Ω_f extending between Γ_i and Γ_∞. Further, in order to define an exact restriction of the unbounded-domain problem to $\Omega_s \cup \Omega_f$, an exact non-reflecting boundary condition should be imposed on Γ_∞. This is done by imposing the Dirichlet-to-Neumann (DtN) condition [4],

$$\nabla p \cdot \mathbf{n} \;=\; S(p) := \sum_{n=0}^{\infty} \alpha_n \frac{k_0 H_n'(k_0 R)}{2\pi H_n(k_0 R)} \int_0^{2\pi} \cos n(\theta - \hat{\theta})\, p(R, \hat{\theta})\, \mathrm{d}\hat{\theta} \qquad \text{on } \Gamma_\infty, \tag{8}$$

in place of (5). Here, R denotes the radius of the truncation boundary Γ_∞, $0 \le \theta < 2\pi$ is the angular coordinate, $H_n(\cdot)$ denotes the n-th order Hankel function of the first kind, $H_n'(\cdot)$ is its derivative, $\alpha_0 := 1$, and $\alpha_n := 2$ if $n \ge 1$.

Finite-element equations

First, we define the appropriate finite-element spaces. As the space of trial solutions and weighting functions for the structural displacements, modeled with first-order shear deformations, we use $\mathcal{S}_{\mathbf{u}} := \{\, \mathbf{u} \mid u_i \in H^1(\Omega_s),\ u_i = 0 \text{ on } \Gamma_i,\ i = 1, 2, \ldots, n_{\text{dof}} \,\}$. As the space for the acoustic pressure we use $\mathcal{S}_p := \{\, p \mid p \in H^1(\Omega_f),\ p = 0 \text{ on } \Gamma_i \,\}$. The variational or weak form of the problem is then as follows: find $\mathbf{u} \in \mathcal{S}_{\mathbf{u}}$ and $p \in \mathcal{S}_p$ such that, for all $\hat{\mathbf{u}} \in \mathcal{S}_{\mathbf{u}}$ and $\hat{p} \in \mathcal{S}_p$,

$$A_f(\hat{p}, p) + (\hat{p}\mathbf{n}, \rho_0\omega^2\mathbf{u})_{\Gamma_i} + (\hat{p}, S(p))_{\Gamma_\infty} = 0, \tag{9}$$

$$A_s(\hat{\mathbf{u}}, \mathbf{u}) + (\hat{\mathbf{u}}, \mathbf{n}p)_{\Gamma_i} = L_s(\hat{\mathbf{u}}). \tag{10}$$

Here, $A_f(\hat{p}, p) := (\nabla\hat{p}, \nabla p)_{\Omega_f} - k_0^2(\hat{p}, p)_{\Omega_f}$ and $L_s(\hat{\mathbf{u}}) := (\hat{\mathbf{u}}, \mathbf{f}_s)_{\Omega_s}$. Also, for the case when the structure is modeled as a Timoshenko beam (as in the numerical examples below), $n_{\text{dof}} = 2$ and

$$A_s(\hat{\mathbf{u}}, \mathbf{u}) := \int_{\Omega_s} \left[\overline{(\hat{u}_{,x} - \hat{v})}\kappa GA\,(u_{,x} - v) + \overline{\hat{v}_{,x}}EIv_{,x} - \rho\omega^2\left(\overline{\hat{u}}Au + \overline{\hat{v}}Iv\right)\right] \mathrm{d}x, \tag{11}$$

where $\mathbf{u} = [\, u \quad v \,]^{\mathrm{T}}$, E and G are elastic and shear moduli, κ is the shear-correction factor, ρ is the structural density, and A and I denote the beam's area and moment of inertia. By choosing $\mathcal{S}_{\mathbf{u}}^h \subset \mathcal{S}_{\mathbf{u}}$ and $\mathcal{S}_p^h \subset \mathcal{S}_p$ to be finite-dimensional subspaces of continuous piecewise-linear polynomials, the Galerkin form of the problem (9)–(10) becomes: find $\mathbf{u}^h \in \mathcal{S}_{\mathbf{u}}^h$ and $p^h \in \mathcal{S}_p^h$ such that, for all $\hat{\mathbf{u}}^h \in \mathcal{S}_{\mathbf{u}}^h$ and $\hat{p}^h \in \mathcal{S}_p^h$,

$$A_s(\hat{\mathbf{u}}^h, \mathbf{u}^h) + (\hat{\mathbf{u}}^h, \mathbf{n}p^h)_{\Gamma_i} = L_s(\hat{\mathbf{u}}^h), \tag{12}$$

$$A_f(\hat{p}^h, p^h) + (\hat{p}^h\mathbf{n}, \rho_0\omega^2\mathbf{u}^h)_{\Gamma_i} + (\hat{p}^h, S(p^h))_{\Gamma_\infty} = 0. \tag{13}$$

Finally, introducing the finite-element nodal basis functions in (12)–(13) in the usual way (see, e.g., [3, Chapter 2]), we obtain the matrix problem

$$\mathbf{K}\,\mathbf{d} = \mathbf{f}, \tag{14}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_s - \omega^2\mathbf{M}_s & \mathbf{C}^T \\ \mathbf{C} & \frac{1}{\rho_0\omega^2}\left(\mathbf{K}_f - k_0^2\mathbf{M}_f + \mathbf{K}_{\text{DtN}}\right) \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{0} \end{bmatrix}. \tag{15}$$

Here, $\mathbf{u} \in \mathbb{C}^{N_s}$ and $\mathbf{p} \in \mathbb{C}^{N_f}$ are the structural and fluid degree-of-freedom vectors, $\mathbf{f}_1 \in \mathbb{C}^{N_s}$ arises from $L_s(\cdot)$ in (12), and $N = N_s + N_f$ is the total number of degrees of freedom. The structural stiffness and mass matrices, $\mathbf{K}_s$ and $\mathbf{M}_s$, arise from (11), the fluid matrices $\mathbf{K}_f$ and $\mathbf{M}_f$ from the operator $A_f(\cdot, \cdot)$ in (13), and $\mathbf{C}$ and $\mathbf{K}_{\text{DtN}}$ from inner products defined in (13). Note that, in order to obtain a complex symmetric matrix $\mathbf{K}$, the fluid equations resulting from (9) were scaled with $1/(\rho_0\omega^2)$ in (15).

Multiple right-hand sides

Frequently, the matrix problem (14) has to be solved repeatedly for different right-hand sides $\mathbf{f}^{(j)}$, $j = 1, 2, \ldots, m$, but with the same matrix $\mathbf{K}$. This is the case when radiation due to several load patterns $\mathbf{f}_s$ is to be studied independently. Multiple right-hand sides also arise in scattering problems. If p_{inc} is the known pressure field due to an incident plane wave and p_{sc} is the unknown scattered field, then the governing equations can be rewritten in terms of p_{sc} and $\mathbf{u}$, whereby, instead of (9)–(10), the corresponding variational equations become

$$A_f(\hat{p}_{\text{sc}}, p_{\text{sc}}) + (\hat{p}_{\text{sc}}\mathbf{n}, \rho_0\omega^2\mathbf{u})_{\Gamma_i} + (\hat{p}_{\text{sc}}, S(p_{\text{sc}}))_{\Gamma_\infty} = -(\hat{p}_{\text{sc}}, \nabla p_{\text{inc}} \cdot \mathbf{n})_{\Gamma_i}, \tag{16}$$

$$A_s(\hat{\mathbf{u}}, \mathbf{u}) + (\hat{\mathbf{u}}, \mathbf{n}p_{\text{sc}})_{\Gamma_i} = L_s(\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathbf{n}p_{\text{inc}})_{\Gamma_i}. \tag{17}$$

The additional terms on the right-hand sides of (16)–(17) give rise to multiple vectors $\mathbf{f}^{(j)}$, each corresponding to a different incident plane wave. In both radiation and scattering problems, all right-hand sides $\mathbf{f}^{(j)}$ are available simultaneously, and hence solving the m systems,

$$\mathbf{K}\,\mathbf{d}^{(j)} = \mathbf{f}^{(j)}, \quad j = 1, 2, \ldots, m, \tag{18}$$

is equivalent to solving the block system, $\mathbf{K}\,\mathbf{D} = \mathbf{F}$, where $\mathbf{D} = \begin{bmatrix}\mathbf{d}^{(1)} & \mathbf{d}^{(2)} & \cdots & \mathbf{d}^{(m)}\end{bmatrix}$ and $\mathbf{F} = \begin{bmatrix}\mathbf{f}^{(1)} & \mathbf{f}^{(2)} & \cdots & \mathbf{f}^{(m)}\end{bmatrix}$. The BL-QMR algorithm is an iterative method tailored to the solution of block systems $\mathbf{K}\,\mathbf{D} = \mathbf{F}$.

THE BL-QMR METHOD

The BL-QMR method proposed in [2] is a block Krylov-subspace algorithm for simultaneous solution of multiple systems (1), where $\mathbf{A} \in \mathbb{C}^{N \times N}$ is in general non-Hermitian. In this section, we review the basic structure of BL-QMR; for further details, we refer the reader to [2].

Consider the solution of multiple linear systems (1), which may be related to (18) via a suitable preconditioner, with all the right-hand sides $\mathbf{b}^{(j)}$, $j = 1, 2, \ldots, m$, being available simultaneously. Then (1) is equivalent to the block system of linear equations

$$\mathbf{A}\,\mathbf{X} = \mathbf{B}, \tag{19}$$

where $\mathbf{B} = \begin{bmatrix}\mathbf{b}^{(1)} & \mathbf{b}^{(2)} & \cdots & \mathbf{b}^{(m)}\end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(m)}\end{bmatrix}$. Let $\mathbf{X}_0 \in \mathbb{C}^{N \times m}$ be an arbitrary initial guess for $\mathbf{A}^{-1}\mathbf{B}$, and set $\mathbf{R}_0 := \mathbf{B} - \mathbf{A}\,\mathbf{X}_0$. The BL-QMR method constructs approximate solutions of (19) that are of the form

$$\mathbf{X}_\mu = \begin{bmatrix}\mathbf{x}_\mu^{(1)} & \mathbf{x}_\mu^{(2)} & \cdots & \mathbf{x}_\mu^{(m)}\end{bmatrix} \in \mathbb{C}^{N \times m}, \quad \mu = 1, 2, \ldots, \tag{20}$$

where $\mathbf{x}_\mu^{(j)} \in \mathbf{x}_0^{(j)} + \mathcal{K}_\mu(\mathbf{A}, \mathbf{R}_0)$ for each $j = 1, 2, \ldots, m$. Here, $\mathcal{K}_\mu(\mathbf{A}, \mathbf{R}_0)$ denotes the *μ-th block Krylov subspace*, which is defined as the μ-dimensional subspace of $\mathbb{C}^N$ spanned by the first μ linearly independent columns of the *right block Krylov matrix* $\begin{bmatrix}\mathbf{R}_0 & \mathbf{A}\mathbf{R}_0 & \cdots & \mathbf{A}^{N-1}\mathbf{R}_0\end{bmatrix}$. The BL-QMR method uses a novel Lanczos-type process [1] to generate two sequences of *biorthogonal* basis vectors $\{\mathbf{v}_\mu\}_{\mu \geq 1}$ and $\{\mathbf{w}_\mu\}_{\mu \geq 1}$ for $\mathcal{K}_\mu(\mathbf{A}, \mathbf{R}_0)$ and $\mathcal{K}_\mu(\mathbf{A}^{\mathrm{T}}, \mathbf{L})$, respectively. Here, $\mathbf{L} \in \mathbb{C}^{N \times m}$ is an arbitrarily chosen matrix, different from the zero matrix. More precisely, after μ iterations, the Lanczos-type process has produced vectors such that

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\mu\} = \mathcal{K}_\mu(\mathbf{A}, \mathbf{R}_0) \quad \text{and} \quad \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_\mu\} = \mathcal{K}_\mu(\mathbf{A}^T, \mathbf{L}). \tag{21}$$

An important feature of the Lanczos-type process is that, during the generation of the sequence $\{\mathbf{v}_\mu\}_{\mu \geq 1}$ (respectively $\{\mathbf{w}_\mu\}_{\mu \geq 1}$) linearly dependent columns in the block Krylov matrices arising from $\mathbf{A}$ and $\mathbf{R}_0$ (respectively $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{L}$) are detected easily and then simply deleted. The process of dropping these dependent vectors from the respective basis is called *deflation*. Performing deflation is crucial in order to obtain robust block Krylov-subspace iterations for multiple linear systems. Indeed, ignoring necessary deflation, usually results in stagnation of the iteration.

The recurrence relations used in the Lanczos-type algorithm to generate the sequence $\{\mathbf{v}_\mu\}_{\mu \geq 1}$ can be summarized as follows:

$$\mathbf{A}\,\mathbf{V}_\mu = \mathbf{V}_n\,\mathbf{T}_\mu + \widehat{\mathbf{V}}_\mu^{\mathrm{dl}}, \quad \mu > 0, \tag{22}$$

$$\mathbf{V}_{m_1}\,\boldsymbol{\rho} + \widehat{\mathbf{V}}_0^{\mathrm{dl}} = \mathbf{R}_0. \tag{23}$$

Here, $\mathbf{V}_\mu := \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_\mu\end{bmatrix}$, $\mathbf{T}_\mu$ is an $n \times \mu$ banded matrix containing the recurrence coefficients, $\widehat{\mathbf{V}}_\mu^{\mathrm{dl}}$ consists of mostly zero columns plus the deflated vectors in the $\mathbf{v}$ sequence, and $\boldsymbol{\rho}$ contains the coefficients used to biorthogonalize the initial block $\mathbf{R}_0$. By (21), the iterates (20) can be parameterized as follows:

$$\mathbf{X}_\mu = \mathbf{X}_0 + \mathbf{V}_\mu\,\mathbf{Z}, \quad \mathbf{Z} \in \mathbb{C}^{\mu \times m}. \tag{24}$$

After each deflation in the $\mathbf{v}$ or $\mathbf{w}$ sequence, the size of the current block in the corresponding deflated block Krylov sequence also decreases by one. Consequently, the size of current blocks in the deflated block Krylov sequences may not always be equal, even though initially $\mathbf{R}_0$ and $\mathbf{L}$ had the same number of columns. Additionally, it can be shown that each time a deflation is encountered in the $\mathbf{v}$ sequence, one can also drop one of the linear systems in (19) whose solution can be recovered from the solutions of the remaining multiple linear systems (see [2]). In effect, this is equivalent to dropping one of the columns of $\mathbf{X}_\mu$, $\mathbf{X}_0$, and $\mathbf{Z}$ in (24) from subsequent solution iterations. Let m_{cr} denote the size of the current block in the *deflated* right block Krylov matrix. Also, let $\mathbf{X}_\mu^{\mathrm{cr}}$ denote the current block of iterates containing m_{cr} columns of $\mathbf{X}_\mu$ that were retained after all deflations. Then, we can rewrite (24) as

$$\mathbf{X}_\mu^{\mathrm{cr}} = \mathbf{X}_0^{\mathrm{cr}} + \mathbf{V}_\mu\,\mathbf{Z}, \quad \mathbf{Z} \in \mathbb{C}^{\mu \times m_{\mathrm{cr}}}. \tag{25}$$

From (25), (22) and (23), the residual block $\mathbf{R}_\mu^{\mathrm{cr}}$ corresponding to $\mathbf{X}_\mu^{\mathrm{cr}}$ satisfies

$$\mathbf{R}_\mu^{\mathrm{cr}} = \mathbf{R}_0^{\mathrm{cr}} - \mathbf{V}_n\,\mathbf{T}_\mu\,\mathbf{Z} - \widehat{\mathbf{V}}_\mu^{\mathrm{dl}}\,\mathbf{Z} = \mathbf{V}_n\left(\begin{bmatrix}\boldsymbol{\rho}^{\mathrm{cr}}\\ \mathbf{0}\end{bmatrix} - \mathbf{T}_\mu\,\mathbf{Z}\right) - \widehat{\mathbf{V}}_\mu^{\mathrm{dl}}\,\mathbf{Z}. \tag{26}$$

Here, the matrix $\boldsymbol{\rho}^{\mathrm{cr}} \in \mathbb{C}^{m_1 \times m_{\mathrm{cr}}}$ contains the columns of $\boldsymbol{\rho}$ in (23) that correspond to columns of $\mathbf{X}_\mu^{\mathrm{cr}}$. Ideally, one would like to choose $\mathbf{Z}$ in (26) such that $\|\mathbf{R}_\mu\|_2$ is minimal. However, in general, this would result in an algorithm with computational work and storage per μ-th iteration that increases with μ. Instead, in order to keep work and storage per iteration low and constant, in [2], we proposed to choose $\mathbf{Z} = \mathbf{Z}_\mu \in \mathbb{C}^{\mu \times m_{\mathrm{cr}}}$ as the solution of the matrix least-squares problem

$$\left\|\begin{bmatrix}\boldsymbol{\rho}^{\mathrm{cr}}\\ \mathbf{0}\end{bmatrix} - \mathbf{T}_\mu\,\mathbf{Z}_\mu\right\|_2 = \min_{\mathbf{Z}\in\mathbb{C}^{\mu\times m_{\mathrm{cr}}}} \left\|\begin{bmatrix}\boldsymbol{\rho}^{\mathrm{cr}}\\ \mathbf{0}\end{bmatrix} - \mathbf{T}_\mu\,\mathbf{Z}\right\|_2. \tag{27}$$

Note that $\mathbf{Z} = \mathbf{Z}_\mu$ results in the minimization of the bracketed term in (26), and therefore, the iterates $\mathbf{X}_\mu$ are characterized by a *quasi-minimization* of the residuals. The resulting iterative scheme for solving multiple linear systems (19) is the BL-QMR method. We refer the reader to [2] for further details of the BL-QMR method.
Finally, if $\mathbf{A}$ is complex symmetric, then the sequence $\{\,\mathbf{w}_\mu\,\}_{\mu\geq 1}$ can be eliminated completely, and the resulting complex-symmetric variant of BL-QMR requires only half the computational work and storage of BL-QMR for general non-Hermitian $\mathbf{A}$; see [5] for details.

NUMERICAL EXAMPLES

We now present two numerical examples. For both, we used the complex-symmetric variant of BL-QMR with SSOR preconditioning, and we stopped the iterations as soon as the relative residuals have converged below $\mathtt{ctol} = 10^{-6}$.
As our first example, we consider the scattering of plane waves from a rigid cylinder with an aspect ratio of $l/d = 8$, $d = 1$, and enclosed by an artificial boundary Γ_∞ of radius $R = l$. For rigid scattering, the boundary-value problem described earlier simplifies to equations (3), (4), and (8) only. The resulting matrix problem simplifies to the form $\left[\mathbf{K}_f - k_0^2\,\mathbf{M}_f + \mathbf{K}_{\mathrm{DtN}}\right]\,\mathbf{p} = \mathbf{f}$. Here, $\mathbf{K}_f$, $\mathbf{M}_f$, and $\mathbf{K}_{\mathrm{DtN}}$ are $N_f \times N_f$ matrices as defined in (15), and $\mathbf{f}$ is defined by inserting the incident pressure field due to a unit-amplitude plane wave,

$$p_{\mathrm{inc}}(\mathbf{x}) = \exp(\mathrm{i}\,k(x\cos\theta_j + y\sin\theta_j)), \tag{28}$$

into the right-hand side of (16). Further, multiple right-hand sides $\mathbf{f}^{(j)}$, $j = 1, 2, \ldots, m$, result due to multiple angles of incidence of the plane wave $\theta_j = (j-1)\,\Delta\theta$ in (28).
Table 1 shows the iteration counts and the number of deflations (in parenthesis) performed for block sizes, m, between 1 and 40 using the BL-QMR method. Note that as m increases,

the BL-QMR method requires fewer and fewer number of iterations per right-hand side. Also note that as m increases, progressively larger number of deflations are performed; these deflations were necessary to ensure linear independence of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m_1}$ in the initial block. We stress that it is crucial to strictly enforce the deflation of linearly dependent and almost linearly dependent vectors in block solution methods. For example, if we do not enforce deflation in the tests shown in Table 1, then for the case $m = 10$ convergence is achieved only after 316 iterations and for $m = 20$ no convergence is achieved even after 900 iteration.

Table 1: Iteration counts for convergence of BL-QMR for rigid scattering

Number of RHS (m)	1	5	10	20	40
Number of iterations	41	98	128	169	366
Number of deflations	0	0	3	11	27

As our second example, we consider radiation from a fluid-loaded elastic structure—a box frame consisting of four interconnected steel plates each of thickness 0.15cm arranged in a rectangular configuration of length $l = 6$cm and width $d = 1.0$cm—that is excited by an externally-applied point load. Longitudinal motion, excluded in (11), is also included to ensure continuity of displacement and rotations. We set $k_0 = 1.0\text{cm}^{-1}$ and use 6720 bilinear quadrilateral elements for discretization, so that $N_s = 480$, $N_f = 6560$ and $\mathbf{K}$ in (15) has size $N = 7040$. We consider seven right-hand sides arising from point forces of similar magnitude and direction applied on adjacent node locations in the finite-element mesh.

Table 2: Iteration counts for convergence of BL-QMR for fluid-loaded elastic radiator

Number of RHS (m)	1	2	3	4	5	6	7
Number of iterations	557	517	521	522	524	533	524
Number of deflations	0	0	1	2	3	4	5

The iteration counts and the number of deflations required by preconditioned BL-QMR (see Table 2) show that the simultaneous solution of these systems is obtained very efficiently—while in most cases the iteration counts increase only modestly as m increases, in some cases the number of iterations actually decreases when additional systems are solved simultaneously! The explanation of this phenomenon is that BL-QMR becomes less prone to loss of biorthogonality as the block size m increases. More precisely, in the underlying Lanczos-type algorithm, biorthogonality of the Lanczos vectors is explicitly enforced only among segments of $2m$ consecutive vectors. The remaining biorthogonality conditions are satisfied in exact arithmetic, but are gradually lost in finite-precision arithmetic due to round-off. While this loss of biorthogonality does not prevent convergence of the BL-QMR method, it typically increases the convergence counts. Consequently, as m increases, more biorthogonality is enforced explicitly and, for BL-QMR, convergence counts per right-hand side can actually decrease.

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