Stable and Passive Reduced-Order Models Based on Partial Padé Approximation Via the Lanczos Process*

Zhaojun Bai, Peter Feldmann, and Roland W. Freund
Bell Laboratories
Lucent Technologies
700 Mountain Avenue
Murray Hill, NJ 07974–0636, USA

Abstract

This paper describes the use of partial Padé approximation to generate stable and passive reduced-order models of linear circuits. For similarly-sized models, partial Padé-based reduced-oreder modeling has superior moment-matching capabilities than competing techniques based on the Arnoldi process. The paper introduces PVLπ, an algorithm for computing partial Padé-based reduced-order models via the Lanczos process. The effectiveness of this modeling methodology is illustrated by numerical examples.

1 Introduction

Electronic circuits often contain large linear subnetworks of passive components. Such subnetworks may represent interconnect automatically extracted from layout as large RLC networks, models of IC packages, models of wireless propagation channels, etc.

We are often interested to model such linear subnetworks by reduced-order models. Ideally the reduced-order models would produce a good approximation of the input-output behavior of the original linear system, at least in a limited domain of interest, e.g., a frequency range.

The transfer function of a lumped-element RLC circuit is always a rational polynomial in the Laplace variable, s. Most model reduction techniques concentrate in a the approximation of the transfer function by another rational polynomial in s, hopefully of a much smaller order. One very useful and widely studied approximation technique by a rational polynomial, is the Padé approximation. A Padè approximation of type (m, n) is defined to be the ratio of polynomials of degree m and n, respectively, whose Taylor expansion about a given point coincides with the that of the original function in the first $m + n + 1$ terms. For the approximation of circuit transfer functions, it is very natural to employ Padé approximations of type $(n - 1, n)$.

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The Padé approximation is optimal in the sense that it matches the maximum number of Taylor coefficients allowed by the available degrees of freedom. These Taylor coefficients can be shown to also represent the moments of the circuit's time-domain impulse response. Simultaneously, it captures the dominant poles of the transfer functions. Since all the $2n$ degrees of freedom in the approximation are consumed to match the $2n$ Taylor coefficients, it can be shown that the Padé approximation is uniquely defined for any order. Moreover, Padé approximations of circuit transfer functions can be computed robustly and efficiently via Lanczos-type algorithms.

In many applications, in particular those related to VLSI interconnect, the Padé approximation is used as a substitute for the full-blown original model in higher level simulations. In such applications it is very important for the reduced-order model to maintain the stability and passivity properties of the original circuit. In [11, 12], we show that Padé approximation based reduced-order model has been proven to be stable and passive for RC, RL, and LC circuits. However, such guarantee cannot be extended to general RLC circuits. As a consequence, a number of alternative approximation techniques, based mainly on the Arnoldi process [3, 22, 17] and congruence transforms [15] have been proposed recently. Assuming a particular formulation of the circuit equations, these algorithms may guarantee the stability and the passivity of the reduced-order models, but will produce rational approximations that match at most half the number of moments for any given order of approximation. However, the resulting rational approximations are no longer uniquely determined by the original transfer functions, and they also require a very specific formulation of the circuit equations to guarantee stability and passivity. For example, starting from a system of nodal equations would result in a different reduced-order model than starting from the sparse tableau equations of the same circuit. In fact, in Section 2, we will show an example of a simple RLC circuit for which the Arnoldi-based reduced-order model is unstable if a certain formulation of the circuit equations is used. In contrast, the Padé approximation will yield identical results, independent of the chosen formulation of the circuit equations.

In most applications, the reduced-order models are used repeatedly, in numerous higher level simulations. In order to achieve maximum computational savings they should be as small as dictated by the accuracy requirements. It has been observed that in the Padé approximations of RLC circuits the violation of stability and passivity conditions are due to a small number of poles and zeros. Therefore, in this paper we explore a method that computes the partial Padé approximation: a Padé-type approximation for which a number of poles and zeros can be prescribed. The partial Padé approximation can be used to produce stable and/or passive reduced-order models and will match a number of moments equal to the remaining number of degrees of freedom, typically much larger than the number matched by competing methods. The partial Padé approximation is uniquely defined by the circuit transfer function and the prescribed singularities and does not depend on the specific formulation of circuit equations. We also introduce PVLπ, an algorithm that computes a partial Padé approximation, by post-processing the Padé approximation computed by the PVL algorithm.

The remainder of the paper is organized as follows. In Section 2, we review several Krylov subspace based reduced-order modeling techniques, and in Section 3 we discuss the partial Padé approximation. Section 4 describes PVLπ, an algorithm that computes the stable and passive partial Padé based reduced-order model from an initial unstable or non-passive Padé result of the PVL algorithm. In Section 5, we report the results of numerical experiments for two circuit examples. Finally, in Section 6, we make some concluding remarks.

2 Krylov Subspace-Based Reduced-Order Modeling

In this section, we briefly review reduced-order modeling techniques based on Krylov subspaces.

2.1 Circuit Transfer Functions

A lumped, linear(ized), time-invariant circuit can be described by the following system of first-order algebraic differential equations:

$$\begin{aligned} \mathbf{C}\,\dot{\mathbf{x}}(t) &= -\mathbf{G}\,\mathbf{x}(t) + \mathbf{b}\,u(t), \\ y(t) &= \mathbf{c}^{\mathrm{T}}\,\mathbf{x}(t). \end{aligned} \tag{1}$$

Here, $\mathbf{C}$ and $\mathbf{G}$ are $N \times N$ matrices and represent the contributions of memory elements (such as capacitors and inductors) and memoryless elements (such as resistors), respectively, and $\mathbf{x}(t)$ is a N-dimensional state vector representing the circuit variables. The functions $u(t)$ and $y(t)$ represent the excitation and the output of interest, respectively.

By applying the Laplace transform to the system (1), assuming zero initial condition, we obtain

$$\begin{aligned} s\,\mathbf{C}\,\mathbf{X}(s) &= -\mathbf{G}\,\mathbf{X}(s) + \mathbf{b}\,U(s), \\ Y(s) &= \mathbf{c}^{\mathrm{T}}\,\mathbf{X}(s). \end{aligned}$$

Here $\mathbf{X}(s)$, $U(s)$ and $Y(s)$ denote the Laplace transforms of $\mathbf{x}(t)$, $u(t)$ and $y(t)$, respectively. It follows that the input $U(s)$ and output $Y(s)$ are related by the *transfer function*

$$H(s) = \mathbf{c}^{\mathrm{T}}\,(\mathbf{G} + s\,\mathbf{C})^{-1}\,\mathbf{b}. \tag{2}$$

Let $s_0 \in \mathbb{R}$ be a real expansion point such that the matrix $\mathbf{G} + s_0\,\mathbf{C}$ is nonsingular. Using any factorization of $\mathbf{G} + s_0\,\mathbf{C}$ of the form

$$\mathbf{G} + s_0\,\mathbf{C} = \mathcal{L} \cdot \mathcal{U}, \quad \mathcal{L}, \mathcal{U} \in \mathbb{R}^{N \times N}, \tag{3}$$

we now rewrite the transfer function $H(s)$ as follows:

$$H(s) = \mathbf{l}^{\mathrm{T}}\,(\mathbf{I} - (s - s_0)\,\mathbf{A})^{-1}\,\mathbf{r} \tag{4}$$

where

$$\mathbf{A} := -\mathcal{L}^{-1}\,\mathbf{C}\,\mathcal{U}^{-1}, \quad \mathbf{r} := \mathcal{L}^{-1}\mathbf{b}, \quad \mathbf{l} := \mathcal{U}^{-\mathrm{T}}\mathbf{c}.$$

For example, in (3), we can choose an LU factorization of $\mathbf{G} + s_0\,\mathbf{C}$, or even the trivial factorization $\mathcal{L} = \mathbf{G} + s_0\mathbf{C}$ and $\mathcal{U} = \mathbf{I}$.

By expanding the transfer function (4) about s_0 and setting $\sigma = s - s_0$, we obtain the representation

$$H(s_0 + \sigma) = \sum_{j=0}^{\infty} m_j \sigma^j, \quad \text{where} \quad m_j = \mathbf{l}^{\mathrm{T}}\,\mathbf{A}^j\,\mathbf{r} \quad \text{for all} \quad j. \tag{5}$$

2.2 Padé Approximation and Krylov Subspaces

Note that H is a rational function. In general, H is of type $(N-1, N)$, i.e., its numerator and denominator polynomials are of degree $N-1$ and N, respectively. This suggests to base reduced-order modeling on approximate transfer functions H_n of the form

$$H_n(s_0+\sigma) = \frac{\psi_{n-1}(\sigma)}{\varphi_n(\sigma)}, \tag{6}$$

where ψ_{n-1} and φ_n are polynomials of degree at most $n-1$ and n, respectively, and $n \ll N$. The function (6) is called an *n-th Padé approximant* if the polynomials ψ_{n-1} and φ_n are chosen such that the expansion of H_n about s_0 agrees with (5) in its first $2n$ terms, i.e.,

$$H(s_0+\sigma) = H_n(s_0+\sigma) + \mathcal{O}(\sigma^{2n}), \tag{7}$$

see, e.g., [5].

The AWE method [18, 19] generates n-th Padé approximants by directly computing the coefficients of the polynomials ψ_{n-1} and φ_n in (6) from the first $2n$ *moments*, $m_0, m_1, \ldots, m_{2n-1}$ in (5). However, generating H_n directly from the moments is a highly unstable numerical procedures, and therefore, only of limited use in practical computations; see, e.g., the examples in [8, 9].

Fortunately, the explicit use of moments can be avoided by computing H_n by means of suitable basis vectors for the *Krylov subspaces*

$$\begin{aligned} \mathcal{K}_n(\mathbf{A},\mathbf{r}) &= \operatorname{span}\{\, \mathbf{r}, \mathbf{A}\,\mathbf{r}, \mathbf{A}^2\,\mathbf{r}, \ldots, \mathbf{A}^{n-1}\,\mathbf{r} \,\} \\ \text{and} \quad \mathcal{K}_n(\mathbf{A}^{\mathrm{T}},\mathbf{l}) &= \operatorname{span}\{\, \mathbf{l}, \mathbf{A}^{\mathrm{T}}\,\mathbf{l}, \left(\mathbf{A}^{\mathrm{T}}\right)^2\mathbf{l}, \ldots, \left(\mathbf{A}^{\mathrm{T}}\right)^{n-1}\,\mathbf{l} \,\}. \end{aligned} \tag{8}$$

More precisely, one generates two sequences of vectors

$$\{\mathbf{v}_j\}_{j=1}^n \quad \text{and} \quad \{\mathbf{w}_j\}_{j=1}^n \tag{9}$$

such that

$$\begin{aligned} \operatorname{span}\{\, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \,\} &= \mathcal{K}_n(\mathbf{A},\mathbf{r}) \\ \text{and} \quad \operatorname{span}\{\, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n \,\} &= \mathcal{K}_n(\mathbf{A}^{\mathrm{T}},\mathbf{l}). \end{aligned} \tag{10}$$

Note that

$$\begin{aligned} m_{2j} &= \mathbf{l}^{\mathrm{T}}\,\mathbf{A}^{2j}\,\mathbf{r} = \left((\mathbf{A}^{\mathrm{T}})^j\,\mathbf{l}\right)^{\mathrm{T}}\left(\mathbf{A}^j\,\mathbf{r}\right) \\ \text{and} \quad m_{2j+1} &= \mathbf{l}^{\mathrm{T}}\,\mathbf{A}^{2j+1}\,\mathbf{r} = \left((\mathbf{A}^{\mathrm{T}})^j\,\mathbf{l}\right)^{\mathrm{T}}\,\mathbf{A}\,\left(\mathbf{A}^j\,\mathbf{r}\right). \end{aligned} \tag{11}$$

By (11) and (10), the information contained in the moments $m_0, m_1, \ldots, m_{2n-1}$ is also contained in the quantities

$$\mathbf{w}_j^{\mathrm{T}}\,\mathbf{v}_j \quad \text{and} \quad \mathbf{w}_j^{\mathrm{T}}\,\mathbf{A}\,\mathbf{v}_j, \quad j = 1, 2, \ldots, n. \tag{12}$$

Hence the point of using the basis vectors (10) is that they allow to obtain all necessary "moment information" for constructing H_n from (12), without ever explicitly forming the moments.

2.3 The Lanczos-Based Approach

The Lanczos process [16] constructs the basis vectors (9) so that they are *biorthogonal*, i.e.,

$$\mathbf{w}_j^{\mathrm{T}}\,\mathbf{v}_k = 0 \quad \text{for all} \quad j \neq k, \quad j,k = 1,2,\ldots,n. \tag{13}$$

This can be done by means of simple three-term recurrences. The coefficients of these recurrences form a tridiagonal matrix $\mathbf{T}_n$ from which one can directly obtain the n-th Padé approximant H_n. More precisely, we have

$$H_n(s_0+\sigma) = \left(\mathbf{l}^{\mathrm{T}}\,\mathbf{r}\right)\,\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{e}_1, \tag{14}$$

see, e.g., [14]. The relation (13) is the basis of the PVL described in [8, 9]. The use of the Lanczos process to compute H_n was also advocated in [13]. We remark that the matrix $\mathbf{T}_n$ can also be expressed in terms of the vectors (9). Let

$$\mathbf{V}_n = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} \quad \text{and} \quad \mathbf{W}_n = \begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n\end{bmatrix} \tag{15}$$

denote the $N \times n$ matrices whose columns are just the vectors (9). Then, the matrix $\mathbf{T}_n$ is given by

$$\mathbf{T}_n = \left(\mathbf{W}_n^{\mathrm{T}}\,\mathbf{V}_n\right)^{-1}\mathbf{W}_n^{\mathrm{T}}\,\mathbf{A}\,\mathbf{V}_n. \tag{16}$$

Note that, by (13) and (15), the matrix $\mathbf{W}_n^{\mathrm{T}}\,\mathbf{V}_n$ in (16) is diagonal.

2.4 Arnoldi-Based Approaches

Instead of two vector sequences (9), the Arnoldi process [3] (applied to $\mathbf{A}$ and $\mathbf{r}$) produces only one sequence $\{\mathbf{v}_j\}_{j=1}^n$ that satisfies the first relation in (10). Moreover, the vectors are constructed to be *orthogonal*, i.e.,

$$\mathbf{v}_j^{\mathrm{T}}\,\mathbf{v}_k = 0 \quad \text{for all} \quad j \neq k, \quad j,k = 1,2,\ldots,n.$$

In contrast to the Lanczos process, the Arnoldi vectors cannot be generated by means of short recurrences. This is reflected in the fact that the matrix of recurrence coefficients, $\mathbf{T}^{(\mathrm{A})}$, is a full upper Hessenberg matrix, instead of a tridiagonal matrix in the Lanczos case. In [22], it is proposed to use the following reduced-order transfer function:

$$H_n^{(\mathrm{A})}(s_0+\sigma) = \|\mathbf{r}\|_2\,\mathbf{l}^{\mathrm{T}}\,\mathbf{V}_n\,\left(\mathbf{I}-\sigma\,\mathbf{T}_n^{(\mathrm{A})}\right)^{-1}\mathbf{e}_1. \tag{17}$$

Here, in analogy to (15), $\mathbf{V}_n$ denotes the matrix whose columns are the Arnoldi vectors $\{\mathbf{v}_j\}_{j=1}^n$.

The function (17) is again of the type (6), however, its expansion about s_0 only matches the first n moments of H, see [22]. Therefore, the $H_n^{(\mathrm{A})}$ is not an n-th Padé approximant. In [22], it is also shown that, for RLC circuits, the reduced-order model based on $H_n^{(\mathrm{A})}$ is stable, provided that a specific formulation of the circuit equations is used. One clear disadvantage of Arnoldi-based approaches is that the resulting reduced-order models depend on the circuit-formulation method, in contrast to Lanczos-based methods that, due to the

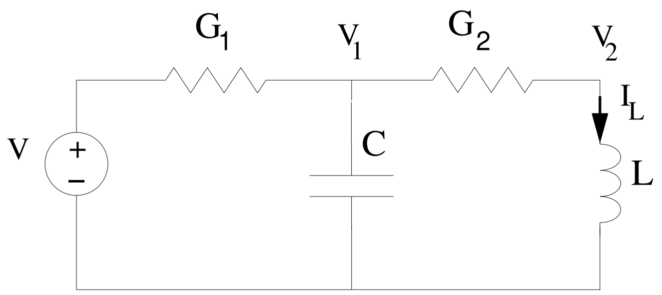

Figure 1: Simple RLC circuit

uniqueness of the underlying n-th Padé approximants, are independent of the formulation of the circuit equations. The following example illustrates this point. We describe a simple RLC circuit and a formulation of the circuit equations for which the reduced-order model $H_1^{(\mathrm{A})}$ is unstable.

The circuit shown in Figure 1 can be described by the equations

$$\begin{aligned} G_1(V-V_1) &= G_2(V_1-V_2)+sCV_1, \\ G_2(V_2-V_1)-I_L &= 0, \\ V_2 &= -sLI_L. \end{aligned} \tag{18}$$

We are interested in the transfer function

$$H(s)=\frac{V_1}{V}.$$

By rewriting (18) in matrix form, it follows that H is given by

$$H(s)=\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{G}+s\,\mathbf{C}\right)^{-1}\left(G_1\mathbf{e}_1\right), \tag{19}$$

where

$$\mathbf{G}=\begin{bmatrix} G_1+G_2 & -G_2 & 0 \\ -G_2 & G_2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{C}=\begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{bmatrix}, \quad \mathbf{e}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

From (19), we obtain a representation of H of the form (4) with expansion point $s_0=0$ and $\mathbf{A}$, $\mathbf{r}$, and $\mathbf{l}$ given by

$$\mathbf{A}=-\mathbf{G}^{-1}\,\mathbf{C}, \quad \mathbf{r}=\mathbf{G}^{-1}\left(G_1\mathbf{e}_1\right), \quad \mathbf{l}=\mathbf{e}_1. \tag{20}$$

Applying one step of the Arnoldi process (to $\mathbf{A}$ and $\mathbf{r}$) results in the reduced-order transfer function

$$H_1^{(\mathrm{A})}(s)=\frac{1}{\left(G_1+G_2\right)\left(1-s\,t_{11}^{(\mathrm{A})}\right)}, \quad \text{where} \quad t_{11}^{(\mathrm{A})}=\frac{\mathbf{r}^{\mathrm{T}}\,\mathbf{A}\,\mathbf{r}}{\mathbf{r}^{\mathrm{T}}\,\mathbf{r}}. \tag{21}$$

We now choose the values $G_1=1$, $G_2=0.1$, $C=1$, and $L=1000$. For these values, we get

$$\mathbf{A}=-\frac{1}{11}\begin{bmatrix} 10 & 0 & 1000 \\ 0 & 0 & 11000 \\ -1 & 0 & 1000 \end{bmatrix}, \quad \mathbf{r}=\frac{1}{11}\begin{bmatrix} 10 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad t_{11}^{(\mathrm{A})}=\frac{7990}{1111}.$$

Hence, the reduced-order transfer function (21) has the unstable pole

$$p_1 = \frac{1}{t_{11}^{(\mathrm{A})}} = \frac{1111}{7990} > 0.$$

In [7], the passivity of Arnoldi-based reduced-order models of the type (17) is investigated. However, in [7], only the necessary condition that a passive system circuit cannot have network zeros in the right half of the complex plane is checked. In general, this condition is not sufficient to guarantee passivity, and one also needs to ensure a positivity condition on the imaginary axis; see, e.g., [2, 24]. In [4], we developed a simple check of this condition that only requires the solution of a certain eigenvalue problem.

Yet another approach to obtaining Arnoldi-based reduced-order models is proposed in [17]. The method consists of first running n steps of the Arnoldi process (applied to $\mathbf{A}$ and $\mathbf{r}$). The result is the matrix $\mathbf{V}_n$ of Arnold vectors. This matrix is then used to project the data $\mathbf{G}$, $\mathbf{C}$, $\mathbf{b}$, and $\mathbf{c}$ from the original representation (2) of H. More precisely, the reduced-order transfer function is defined as follows:

$$H_n^{(\mathrm{A2})}(s) = \left(\mathbf{V}_n^{\mathrm{T}}\,\mathbf{c}\right)^{\mathrm{T}} \left(\mathbf{V}_n^{\mathrm{T}}\,\mathbf{G}\,\mathbf{V}_n + s\,\mathbf{V}_n^{\mathrm{T}}\,\mathbf{C}\,\mathbf{V}_n\right)^{-1} \left(\mathbf{V}_n^{\mathrm{T}}\,\mathbf{b}\right). \tag{22}$$

In [17], it is shown that $H_n^{(\mathrm{A2})}$ matches the first n moments of H, and that $H_n^{(\mathrm{A2})}$ is passive. However, the very same properties can be shown if in (22), instead of the Arnoldi matrix $\mathbf{V}_n$, any matrix $\mathbf{V}_n$ is used that satisfies a relation of the form

$$\mathbf{A}\,\mathbf{V}_n = \mathbf{V}_n\,\mathbf{T} + \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{v} \end{bmatrix},$$

where $\mathbf{T}$ is any $n \times n$ upper Hessenberg matrix and $\mathbf{v}$ is any vector of length N. Therefore, the passivity and moment-matching properties alone do not seem to be enough to appropriately characterize the approximation properties of (2).

Finally, we remark that it is also possible to obtain the n-th Padé approximant H_n by means of the so-called two-sided Arnoldi process [21]. However, this requires two runs of the Arnoldi process: one with $\mathbf{A}$ and $\mathbf{r}$ to generate an orthogonal basis of $\mathcal{K}_n(\mathbf{A}, \mathbf{r})$, and one with $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{l}$ to generate an orthogonal basis of $\mathcal{K}_n(\mathbf{A}^{\mathrm{T}}, \mathbf{l})$. The resulting computational procedure is more expensive than PVL, without having any real advantage over PVL.

3 Partial Padé Approximation

Let $H_n(s)$ be an $(n-1, n)$ Padé approximation to the original transfer function $H(s)$, matching the maximal number of moments: $2n$. The *partial Padé approximation*, proposed by Brezinski [6], is a function of the same form,

$$\widehat{H}_n(s_0 + \sigma) = \left(\mathbf{l}^{\mathrm{T}}\,\mathbf{r}\right) \frac{\widehat{\psi}_{n-1}(\sigma)}{\widehat{\varphi}_n(\sigma)}, \tag{23}$$

where some of the poles and zeros are prescribed. As in the classical Padé approximation the rational function should maximize the number of matched coefficients in a Taylor expansion about s_0. Here the numerator polynomial $\widehat{\psi}_{n-1}$ is of degree $n-1$ with ℓ prescribed roots and the denominator polynomial $\widehat{\varphi}_n$ is of degree n with k prescribed roots. The remaining degrees of freedom, $2n-\ell-k$, could be used to ensure the matching of an equal number of Taylor coefficients, i.e.,

$$H(s_0+\sigma) = \widehat{H}_n(s_0+\sigma) + \mathcal{O}(\sigma^{2n-\ell-k}). \tag{24}$$

In [6], the Taylor expansion of the function to be approximated is explicitly applied to construct a partial Padé approximation. The more restricted problem of prescribing just some of the poles in Padé approximations is widely studied in literature, see for example [5] and the references therein. The common terminology used there is *Padé-type approximations*.

It has been shown that the Padé approximation $H_n(s)$ of an RLC circuit transfer function will occasionally be unstable and/or non-passive despite the fact that RLC circuits are always stable and passive. The instability or the non-passivity of the transfer function $H(n)$ is, in general, caused by a small number of misplaced poles and zeros. The effects of these poles and zeros on the approximate transfer function in the domain of interest must be minimal, once the accuracy of the Padé approximation becomes satisfactory.

We propose using the partial Padé approximation to generate stable and/or passive reduced-order models of circuit transfer functions through post-processing of the Padé approximation originally computed with the PVL algorithm.

Our method consists in: (1), computation of the Padé approximation of the transfer function via a Lanczos process (the PVL algorithm), (2), identification of the misplaced poles and/or zeros that responsible with the instability or the non-passivity of the reduced order model, if any, (3), prescription of alternative locations for these poles and/or zeros, and, (4), computation of the corresponding partial Padé approximation using a novel algorithm. Since we argue that the effect of these poles and zeros on the transfer function in the domain of interest must be small, we expect little or no perturbation to the already converged poles and/or zeros. (5), finally, we check the stability and/or passivity of the result using the eigenvalue based criterion described in [4].

In the rest of this paper, we shall propose a novel method for computing a partial Padé approximation and demonstrate via numerical experiments its usefulness for enforcing stability and passivity in reduced-order models. The significant novelty in our approach for constructing a partial Padé approximation consists of the use of a matrix update technique on the tridiagonal matrix obtained by a Lanczos process instead of using moments explicitly as proposed by Brezinski [6]. The resulting partial Padé approximation obtained via the Lanczos process and post-processing is significantly superior in terms of both numerical stability and computational efficiency.

4 Postprocessing of PVL: The PVLπ Algorithm

As remarked in the previous section, instead of using moments explicitly for the computation of the partial Padé approximation we prefer to post-process the matrix $\mathbf{T}_n$ generated by the PVL algorithm. The partial Padé approximation problem is therefore recast as the problem of finding an updating matrix $\mathbf{F}_n$ such that some of the eigenvalues of the matrices $\widehat{\mathbf{T}}_n = \mathbf{T}_n + \mathbf{F}_n$ and $\widehat{\mathbf{T}}'_n = \mathbf{T}'_n + \mathbf{F}'_n$ are the prescribed values. Here, for any given matrix $\mathbf{X}$, $\mathbf{X}'$ denotes the matrix obtained by deleting the first row and column of $\mathbf{X}$. Recall that the zeros of the reduced-order transfer function H_n are given by the eigenvalues of $\mathbf{T}'_n$; see, e.g., [9]. For a properly chosen updating matrix $\mathbf{F}_n$, we will have a modified reduced-order transfer function associated with the updated matrix $\widehat{\mathbf{T}}_n$ such that

$$\widehat{H}_n(s_0+\sigma) = \left(\mathbf{l}^{\mathrm{T}}\mathbf{r}\right)\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\widehat{\mathbf{T}}_n\right)^{-1}\mathbf{e}_1 = H(s_0+\sigma)+\mathcal{O}(\sigma^{2n-\ell-k}). \tag{25}$$

Using the formulation of the Lanczos process in terms of polynomials instead of vectors, it can be shown that the updating matrix $\mathbf{F}_n$ must be a rank-one matrix of the form

$$\mathbf{F}_n = \mathbf{x}\mathbf{e}_n^{\mathrm{T}}.$$

Furthermore, if there are a total of m prescribed poles and zeros, where $m = k+\ell \le n$, then the vector $\mathbf{x}$ can be chosen as

$$\mathbf{x} = \begin{bmatrix}\mathbf{0}\\ \mathbf{x}_m\end{bmatrix}.$$

To verify the order of approximation of the modified reduced-order transfer function defined by $\widehat{\mathbf{T}}_n = \mathbf{T}_n + \mathbf{x}\mathbf{e}_n^T$, by the Sherman-Morrison formula, we have

$$\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\widehat{\mathbf{T}}_n\right)^{-1}\mathbf{e}_1 = \mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{e}_1 - \tau\sigma\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{x}\,\mathbf{e}_n^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{e}_1. \tag{26}$$

Here, the scalar $\tau^{-1} = 1-\sigma\mathbf{e}_n^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{x}$. By using the formula

$$\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1} = \frac{\operatorname{adj}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)}{\det\left(\mathbf{I}-\sigma\mathbf{T}_n\right)},$$

where $\operatorname{adj}(\mathbf{I}-\sigma\mathbf{T}_n)$ stands for the adjoint matrix made up of $(n-1)\times(n-1)$ cofactors of $\mathbf{I}-\sigma\mathbf{T}_n$, one can verify that

$$\mathbf{e}_n^{\mathrm{T}}\left(\mathbf{I}-\sigma\mathbf{T}_n\right)^{-1}\mathbf{e}_1 = \mathcal{O}(\sigma^{n-1})$$

and

$$\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\widehat{\mathbf{T}}_n\right)^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{x}_m\end{bmatrix} = \mathcal{O}(\sigma^{n-m}).$$

Therefore, the modified reduced-order transfer function defined by (25) has the approximation order $2n-k-\ell$, as defined in (24). One can also directly verify directly by straightforward calculation that

$$\mathbf{l}^{\mathrm{T}}A^j\mathbf{r} = \left(\mathbf{l}^{\mathrm{T}}\mathbf{r}\right)\mathbf{e}_1^{\mathrm{T}}\mathbf{T}_n^j\mathbf{e}_1 = \left(\mathbf{l}^{\mathrm{T}}\mathbf{r}\right)\mathbf{e}_1^{\mathrm{T}}\widehat{\mathbf{T}}_n^j\mathbf{e}_1$$

for $j = 0,1,2,\ldots,2n-m-1$. This again justifies that the modified reduced-order transfer function $\widehat{H}_n(s_0+\sigma)$ is a partial Padé approximation to the original transfer function $H(s_0+\sigma)$.

The following algorithm computes the desired vector $\mathbf{x}$.

Algorithm 1 (*Computation of the vector* $\mathbf{x}$)
INPUT: $\{\theta_j\}_{j=1}^{k}$ and $\{\tau_j\}_{j=1}^{\ell}$, where $m = k + \ell \leq n$.
OUTPUT: Vector $\mathbf{x}$ such that the matrices

$$\widehat{\mathbf{T}}_n = \mathbf{T}_n + \mathbf{x}\mathbf{e}_n^{\mathrm{T}} \quad \text{and} \quad \widehat{\mathbf{T}}'_n = \mathbf{T}'_n + \mathbf{x}'\mathbf{e}_{n-1}^{\mathrm{T}}$$

have the prescribed eigenvalues $\{\theta_j\}_{j=1}^{k}$ and $\{\tau_j\}_{j=1}^{\ell}$, respectively.

1) Compute the row vector $\boldsymbol{\varphi}(\theta_j)$ for $j = 1, 2, \ldots, k$: if $\theta_j \in \lambda(\mathbf{T}_n)$, then $\boldsymbol{\varphi}(\theta_j)$ is the corresponding left eigenvector, else $\boldsymbol{\varphi}(\theta_j) = \mathbf{e}_n^{\mathrm{T}}(\theta_j\mathbf{I} - \mathbf{T}_n)^{-1}$.
2) Compute the row vector $\boldsymbol{\psi}(\tau_j)$ for $j = 1, 2, \ldots, \ell$: if $\tau_j \in \lambda(\mathbf{T}'_n)$, then $\boldsymbol{\psi}(\tau_j)$ is the corresponding left eigenvector, else $\boldsymbol{\psi}(\tau_j) = \mathbf{e}_{n-1}^{\mathrm{T}}(\tau_j\mathbf{I} - \mathbf{T}'_n)^{-1}$.
3) Compute the last column $\mathbf{R}_k(:, n)$ of the $k \times n$ matrix

$$\mathbf{R}_k = \mathbf{\Lambda}_k\, \mathbf{\Phi}_k - \mathbf{\Phi}_k\, \mathbf{T}_n := \operatorname{diag}(\theta_j)\,[\boldsymbol{\varphi}(\theta_j)] - [\boldsymbol{\varphi}(\theta_j)]\,\mathbf{T}_n.$$

4) Compute the last column $\mathbf{S}_\ell(:, n-1)$ of the $\ell \times (n-1)$ matrix

$$\mathbf{S}_\ell = \mathbf{\Upsilon}_\ell\, \mathbf{\Psi}_\ell - \mathbf{\Psi}_\ell\, \mathbf{T}'_n := \operatorname{diag}(\tau_j)\,[\boldsymbol{\psi}(\tau_j)] - [\boldsymbol{\psi}(\tau_j)]\,\mathbf{T}'_n.$$

5) Solve for $\mathbf{x}_m$:

$$\begin{bmatrix} \mathbf{\Phi}_k(:, n-m+1:n) \\ \mathbf{\Psi}_\ell(:, n-m:n-1) \end{bmatrix} \mathbf{x}_m = \begin{bmatrix} \mathbf{R}_k(:, n) \\ \mathbf{S}_\ell(:, n-1) \end{bmatrix}, \quad \text{if} \quad m < n.$$

Or solve for $\mathbf{x}$:

$$\begin{bmatrix} \mathbf{\Phi}_k \\ \tilde{\mathbf{\Psi}}_\ell \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{R}_k(:, n) \\ \mathbf{S}_\ell(:, n-1) \end{bmatrix}, \quad \text{if} \quad m = n,$$

where $\tilde{\mathbf{\Psi}}_\ell = \begin{bmatrix} \mathbf{0} & \mathbf{\Psi}_\ell \end{bmatrix}$.

Note that $\mathbf{X}(:, j)$ and $\mathbf{X}(:, i:j)$ denote the j-th column and the i-th to j-th columns of the matrix $\mathbf{X}$, respectively. Moreover, $\lambda(\mathbf{X})$ denotes the set of eigenvalues of $\mathbf{X}$. If all matrices and vectors are real, and the prescribed complex eigenvalues θ_j and/or τ_j occur in complex conjugate pairs, if any, then the above algorithm can be modified slightly such that the vector $\mathbf{x}$ (and therefore the modified matrix $\widehat{\mathbf{T}}_n$) stays real.

Two special assignments of poles and zeros are of particular interest. The first case is when all poles are prescribed, i.e., $k = n$. Then we have

$$\mathbf{\Phi}_n\mathbf{x}\mathbf{e}_n^{\mathrm{T}} = \mathbf{R}_n,$$

where $\mathbf{\Phi}_n$ is an $n \times n$ matrix. If $\mathbf{\Phi}_n$ is nonsingular, then the vector $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{\Phi}_n^{-1}\mathbf{R}_n(:, n).$$

In this case, all eigenvalues of the matrix $\widehat{\mathbf{T}}_n = \mathbf{T}_n + \mathbf{x}\mathbf{e}_n^{\mathrm{T}}$ are the prescribed values, which implies that all poles of the modified reduced-order transfer function defined by $\widehat{\mathbf{T}}_n$ are the prescribed values. In practice, if $p_j = s_0 + 1/\lambda_j$ for $j = 1, \ldots n$ are the poles of the reduced-order model. Without loss of generality, assume that all poles are stable, except the last two poles p_{n-1} and p_n are unstable. Let $\theta_j = 1/(p_j - s_0)$ for $j = 1, 2, \ldots, n-2$. Let the unstable poles p_{n-1} and p_n be assigned to the stable poles $\widehat{p}_{n-1}$ and $\widehat{p}_n$, and let $\theta_{n-1} = 1/(\widehat{p}_{n-1} - s_0)$ and $\theta_n = 1/(\widehat{p}_n - s_0)$. Then with all these prescribed poles, the modified reduced-order transfer function $\widehat{H}_n(s_0 + \sigma)$ is guaranteed to be stable.

The second special case is when all zeros are prescribed. In this case,

$$\mathbf{\Psi}_{n-1}\mathbf{x}' \cdot \mathbf{e}_{n-1}^{\mathrm{T}} = \mathbf{S}_{n-1},$$

where $\mathbf{\Psi}_{n-1}$ is an $(n-1) \times (n-1)$ matrix. If it is nonsingular, then we have

$$\mathbf{x}' = \mathbf{\Psi}_{n-1}^{-1}\mathbf{S}_{n-1}(:, n-1).$$

As a result, all eigenvalues of the matrix $\widehat{\mathbf{T}}'_n = \mathbf{T}'_n + \mathbf{x}'\mathbf{e}_{n-1}^{\mathrm{T}}$ are the prescribed values, which means that all zeros of the modified reduced-order transfer function defined by $\widehat{\mathbf{T}}_n$ are the prescribed values. With a similar discussion as for assigning poles, it can be guaranteed that the modified reduced-order model has no zeros in the right half-plane.

With the above algorithm to update partial or complete poles and/or zeros of the reduced-order transfer function, we are ready to present a new algorithm, which generates a partial Padé approximation via the Lanczos process. The new algorithm is based on the PVL algorithm and does post-processing in the presence of unstable poles and right-half-plane zeros of the reduced-order transfer function. The resulting computational procedure is named the PVLπ algorithm.

Algorithm 2 (*Sketch of the* PVLπ *algorithm*)

1) Run n steps of the Lanczos process to obtain the tridiagonal matrix $\mathbf{T}_n$.
2) Compute eigendecompositions of $\mathbf{T}_n = \mathbf{S}_n \mathrm{diag}(\lambda_j)\mathbf{S}_n^{-1}$ and $\mathbf{T}'_n = \mathbf{S}'_n \mathrm{diag}(\lambda'_j)(\mathbf{S}'_n)^{-1}$ and obtain the approximate poles and zeros

$$p_j = s_0 + 1/\lambda_j \quad \text{and} \quad z_j = s_0 + 1/\lambda'_j,$$

respectively.
3) Check stability and/or passivity. If the reduced-order model defined by $\mathbf{T}_n$ is stable and/or passive, then set $\widehat{\mathbf{T}}_n = \mathbf{T}_n$ and go to Step 8).
4) Select partial or complete prescribed poles $\widehat{p}_j$ and/or zeros $\widehat{z}_j$. Set

$$\theta_j = 1/(\widehat{p}_j - s_0) \quad \text{and} \quad \tau_j = 1/(\widehat{z}_j - s_0).$$

5) Use Algorithm 1 to compute the vector $\mathbf{x}$.
6) Compute eigendecompositions of $\widehat{\mathbf{T}}_n = \mathbf{T}_n + \mathbf{x}\mathbf{e}_n^{\mathrm{T}} = \widehat{\mathbf{S}}_n \mathrm{diag}(\widehat{\lambda}_j)\widehat{\mathbf{S}}_n^{-1}$ and $\widehat{\mathbf{T}}'_n = \mathbf{T}'_n + \mathbf{x}'\mathbf{e}_{n-1}^{\mathrm{T}} = \widehat{\mathbf{S}}'_n \mathrm{diag}(\widehat{\lambda}'_j)(\widehat{\mathbf{S}}'_n)^{-1}$ and obtain the modified approximate poles and zeros

$$\widehat{p}_j = s_0 + 1/\widehat{\lambda}_j \quad \text{and} \quad \widehat{z}_j = s_0 + 1/\widehat{\lambda}'_j.$$

7) Check stability and/or passivity. Go to Step 4) if necessary.

8) Compute the poles (zeros) and residues of the modified reduced order model

$$\widehat{H}_n(s_0+\sigma) = \left(\mathbf{l}^{\mathrm{T}}\mathbf{r}\right)\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{I}-\sigma\widehat{\mathbf{T}}_n\right)^{-1}\mathbf{e}_1.$$

Recall from Section 2 that all zeros being in the left half–plane is only a necessary condition for passivity, and that an additional positivity condition needs to be checked. In Algorithm 2, the "passivity check" refers to this necessary condition only.

5 Examples

In this section, we present two circuit examples to demonstrate the effectiveness of the PVLπ method.

5.1 The PEEC Circuit

Our first example is a circuit resulting from the PEEC discretization [20] of an electromagnetic problem. This circuit has been often used as a test problem in the model-reduction literature. The circuit consists of only inductors, capacitors, and inductive couplings, and it is driven by a finite resistance source. We are interested in analyzing the response of this circuit, in this case the current flowing through one of the inductors. The circuit being mainly LC, its poles would lie exactly on the imaginary axis in the absence of the 100Ω source resistance. The source resistance causes the poles to move into the left half of the complex plane, but leaves most of them in the proximity of the imaginary axis. Due to this proximity, during the computation of a reduced-order model, numerical and approximation errors can easily produce unstable poles, i.e., poles lying in the right half of the complex plane. Indeed, running the PVL algorithm (with shift $s_0 = 2\pi 10^9$) for $n = 60$ iterations produces an almost exact transfer function in the 0-5GHz range, as shown in Figure 2, but results in 8 unstable poles and 7 nearly stable poles, see Figure 4.

In order to produce a stable reduced-order model, we use PVLπ to force all the 15 unstable poles, $p_j = \mathrm{Re}(p_j) + \mathbf{i}\,\mathrm{Im}(p_j)$, into the left half-plane. This is done by setting, in Algorithm 1,

$$\theta_j = \frac{1}{\widehat{p}_j - s_0}, \quad \text{where} \quad \widehat{p}_j = -\mathrm{Re}(p_j) + \mathbf{i}\,\mathrm{Im}(p_j),$$

for all 15 unstable poles, while the stable poles are unchanged. Figure 4 shows the poles of the reduced-order transfer function $H_{60}(\mathbf{i}\omega)$ produced by the PVL algorithm, and the modified reduced-order transfer function $\widehat{H}_{60}(\mathbf{i}\omega)$ generated by the PVLπ algorithm. The modified, reduced-order transfer function $\widehat{H}_{60}(s)$ has all its poles in the left half plane and is stable! Figure 3 shows the frequency responses of the exact transfer function $H(\mathbf{i}\omega)$ compared to $\widehat{H}_{60}(\mathbf{i}\omega)$ produced by the PVLπ algorithm with 60 iterations, in the 0-5 GHz range. The match remains as good as before.

5.2 A Package Model

The next application is the analysis of a 64-pin package model used for an RF integrated circuit. Only eight of the package pins carry signal, the rest being either unused or carrying

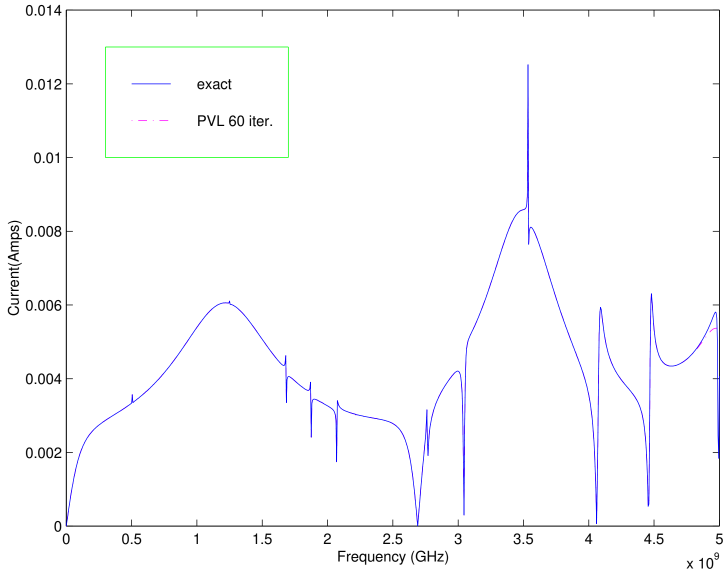

Figure 2: The PEEC circuit transfer function, exact and 60 PVL iterations

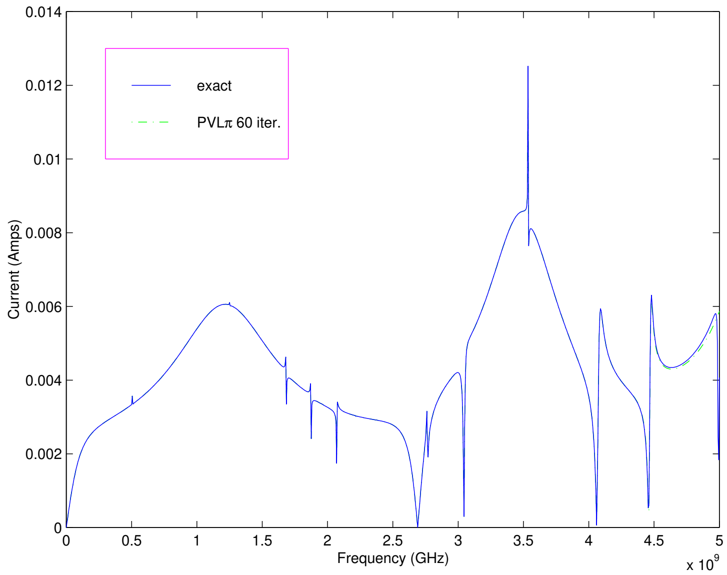

Figure 3: The PEEC circuit transfer function, exact and 60 PVLπ iterations

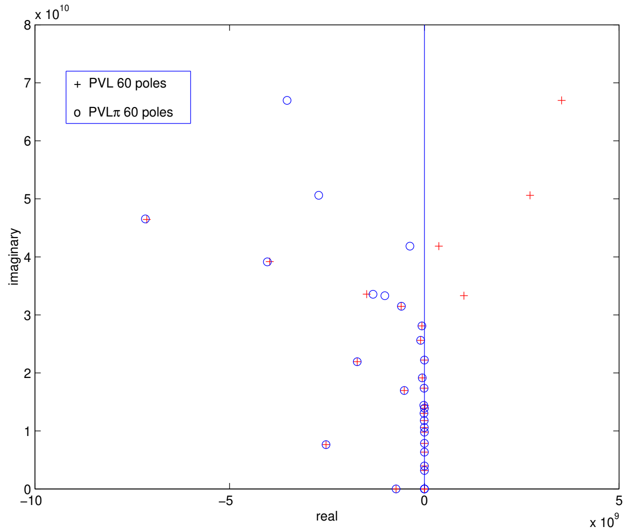

Figure 4: The PEEC circuit, PVL poles and PVLπ poles

supply voltages. The package model is described by approximatively 4000 circuit elements, resistors, capacitors, inductors, and inductive couplings. The size of the nodal circuit matrices for the package is about 2000. Here, we compute the input impedance of one of the pins. After executing the PVL algorithm for $n = 80$ iterations, the resulting reduced-order model has a transfer function H_{80} defined by $\mathbf{T}_{80}$. The approximation H_{80} is sufficiently accurate up to 10GHz, but has 2 poles and 4 zeros violating stability and passivity conditions.

We use PVLπ to force the 2 unstable poles, $p_j = \mathrm{Re}(p_j) + \mathbf{i}\,\mathrm{Im}(p_j)$, and the 4 unstable zeros $z_j = \mathrm{Re}(z_j) + \mathbf{i}\,\mathrm{Im}(z_j)$ into the left half-plane. This is done by setting, in Algorithm 1,

$$\theta_j = \frac{1}{\widehat{p}_j - s_0}, \quad \text{where} \quad \widehat{p}_j = -0.1\,\mathrm{Re}(p_j) + \mathbf{i}\,(10\,\mathrm{Im}(p_j)),$$

for the 2 unstable poles, and

$$\tau_j = \frac{1}{\widehat{z}_j - s_0}, \quad \text{where} \quad \widehat{z}_j = -0.1\,\mathrm{Re}(z_j) + \mathbf{i}\,(10\,\mathrm{Im}(z_j)),$$

for the 4 unstable zeros. The reduced-order model produced by PVLπ has all poles and zeros of the modified reduced order transfer function given by $\widehat{\mathbf{T}}_{80} = \mathbf{T}_{80} + \mathbf{x}\mathbf{e}^{\mathrm{T}}$ in the left complex half-plane, where the vector $\mathbf{x}$ has only $2 + 4 = 6$ nonzero entries at the bottom. By construction, the modified, reduced-order model is stable, and it also satisfies the necessary condition that a passive system cannot have zeros in the right half-plane. To verify passivity, an additional positivity condition has to be satisfied; see, e.g., [2]. In [4], we derived a simple eigenvalue-based test for this positivity condition. Using this test, one easily verifies that the reduced-order model is indeed passive. Figure 5 shows the frequency response of the exact transfer function and the reduced-order transfer function generated by the PVL algorithm. Figure 6 shows the frequency response of the exact transfer function

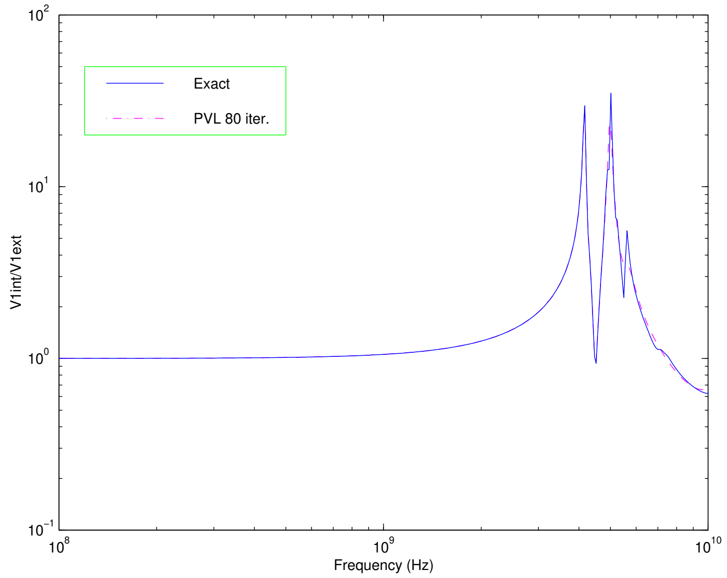

Figure 5: Package: Pin no.1 external to Pin no.1 internal, exact and 80 PVL iterations

and the reduced-order transfer function generated by the PVLπ algorithm. Figure 5 and 6 clearly illustrate that no accuracy has been lost by enforcing stability and passivity by means of our post-processing procedure.

6 Concluding Remarks

In this paper, we have introduced PVLπ, an algorithm for generating a stable and/or passive reduced-order model starting from a PVL-produced model that violates stability and/or passivity conditions. The PVLπ algorithm computes reduced-order models that are characterized by a partial Padé approximation of the original circuit transfer function. This method yields reduced-order models with desired stability and/or passivity properties, while sacrificing less in terms of accuracy than competing methods.

This paper only treated the case of scalar transfer functions. However, similar post-processing techniques can be developed for the MPVL algorithm [10] for matrix-valued transfer functions of multi-input multi-output systems. There the concept of partial matrix Padé approximation is used to produce stable and/or passive reduced-order models. The resulting MPVLπ algorithm will be described elsewhere.

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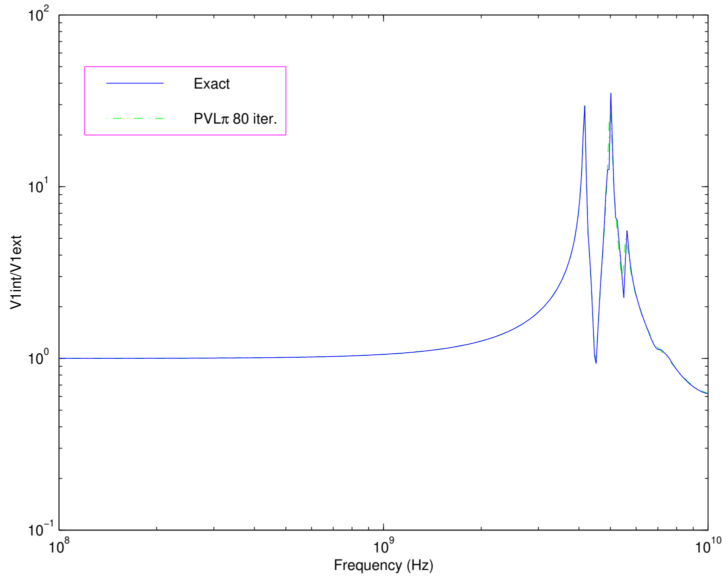

Figure 6: Package: Pin no.1 external to Pin no.1 internal, exact and 80 PVLπ iterations

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