# Eigenvalue-Based Characterization and Test for Positive Realness of Scalar Transfer Functions*

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**Abstract**

An eigenvalue-based characterization of positive realness of transfer functions of single-input single-output time-invariant linear systems is derived. Based on this characterization, we propose efficient computational procedures to determine if a given transfer function is positive real. The input for these eigenvalue-based tests is any given, not necessarily minimal, state-space representation of the linear system. Furthermore, the tests only involve standard matrix computations, such as computing eigenvalues of a matrix or a matrix pencil. Results of numerical experiments with these eigenvalue-based tests for positive realness are reported.

**Key words:** linear system, state-space representation, transfer function, passivity, positive realness

## 1 Introduction

Passivity is an important concept in control and circuit theory. Roughly speaking, a (linear or nonlinear) system is *strictly passive* if it "consumes" energy, and it is *passive* if it does not "deliver" energy. The concept was first used in circuit theory; see, e.g., [3, 17]. For example, networks consisting of only resistors, inductors, and capacitors are passive. Passivity has also

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become a fundamental concept in system and control theory, especially in adaptive control and stability analysis of feedback systems; see, e.g., [5, 9, 12, 23]. In this paper, we only consider time-invariant linear systems. In this case, passivity of a system is equivalent to *positive realness* of the system's transfer function; see, e.g., [3].

The concept of positive real functions was introduced by Brune [10] in 1931. Over the years, a number of procedures for testing the positive realness of a given rational function have been developed. There is a class of procedures for checking positive realness symbolically using Sturm methods, which yields Routh-Hurwitz like algorithms; see, e.g., [17, pp.15–36] and [26]. A major breakthrough came in the early 1960s with the Positive Real (PR) Lemma [19, 20, 21], which connects the positive realness of a scalar rational function with the existence of a positive-definite matrix satisfying certain linear matrix inequalities (LMIs). The result was soon extended to the case of matrix-valued rational functions by Anderson [1]. Today, the PR Lemma is also known as the Kalman-Yakubovich Lemma or Yakubovich-Kalman-Popov-Anderson Lemma; see, e.g., [3, Chapter 6] and [9, Chapter 2]. For a recent survey on the history of the PR Lemma and its applications, we refer the reader to [8]. Variants of the PR Lemma for the characterization and testing of strict positive realness of transfer functions of dissipative systems are described in [2, 18, 24, 27, 30].

In recent years, reduced-order modeling techniques based on Krylov-subspace iterations, such as the Lanczos algorithm, have become popular tools to tackle the large-scale time-invariant linear dynamical systems that arise in the simulation of VLSI circuits; see, e.g., [14, 15] and the references given there. These techniques are mostly applied to very large passive networks, and then it is crucial to check if the reduced-order model produced by the Krylov-subspace technique preserves the passivity of the original network; see, e.g., [7, 13, 15, 22]. These reduced-order models are time-invariant linear systems that are given by state-variable descriptions. Usually, these linear systems have no term that directly couples input to outputs. Moreover, in general, the given state-variable description may not be minimal. The existing numerical procedures for testing the passivity of such reduced-order models are either inadequate or computationally too expensive. More precisely, Routh-Hurwitz like tests require the coefficients of the numerator and denominator polynomials of the rational transfer function. However, working with these polynomial coefficients is inherently numerically unstably for large and even moderately large state-space dimension, and it is preferable to directly work with the given state-variable description; see, e.g., [14]. Testing passivity via the PR Lemma requires the solution of LMIs, which is quite expensive. Under certain assumptions, in particular, for a nonzero directly-coupled input-output term, the LMIs can be reduced to the solution of an algebraic Riccati equation, which in turn can be tackled via an eigendecomposition of a related Hamiltonian matrix; see, e.g., [3, Chapter 6] and [9, Chapter 2]. For reduced-order modeling of VLSI circuits, the assumptions that allow this reduction to a Hamiltonian eigenvalue problem are not satisfied in general. Furthermore, even if this reduction is possible, the solution of an Hamiltonian eigenvalue problem is more expensive than the eigenvalue-based procedures proposed here.

In this paper, we present an eigenvalue-based characterization of positive realness of transfer functions of single-input single-output time-invariant linear systems. Using this characterization, we develop efficient numerical tests of positive realness. These tests only involve standard matrix computations, such as computing eigenvalues of a matrix or a matrix pencil. The input for these eigenvalue-based tests is any given, not necessarily minimal, state-space representation of the linear system.

The remainder of the paper is organized as follows. In Section 2, we collect a few known facts about transfer functions and positive realness. In Section 3, we derive our eigenvalue-based characterization of positive realness. In Section 4, we describe practical numerical test procedures. In Section 5, numerical examples are presented. Finally, we make some concluding remarks in Section 6.

Throughout this article, we use boldface letters to denote vectors and matrices. The identity matrix is denoted by $\mathbf{I}$ and the zero matrix by $\mathbf{0}$; the actual dimensions of $\mathbf{I}$ and $\mathbf{0}$ will usually be apparent from the context. The sets of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ is the real part of $s$. Finally, $\mathbb{C}_+ := \{\, s \in \mathbb{C} \mid \mathrm{Re}(s) > 0 \,\}$ is the open right half of the complex plane.

## 2 Transfer Functions and Positive Realness

We consider single-input single-output time-invariant linear systems given by state-variable descriptions of the form

$$\begin{aligned} \mathbf{E}\,\frac{d}{dt}\mathbf{x}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,u(t), \\ y(t) &= \mathbf{c}^{\mathrm{T}}\,\mathbf{x}(t) + d\,u(t), \end{aligned} \tag{1}$$

where $\mathbf{A}$, $\mathbf{E} \in \mathbb{R}^{n\times n}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^{n}$, and $d \in \mathbb{R}$. For background on linear systems, we refer the reader to [11]. The matrices $\mathbf{A}$ and $\mathbf{E}$ are allowed to be singular, and we only assume that the pencil $\mathbf{A} - s\,\mathbf{E}$ is *regular*, i.e., the matrix $\mathbf{A} - s\,\mathbf{E}$ is singular only for finitely many values of $s \in \mathbb{C}$. Note that (1) is a system of differential-algebraic equations if $\mathbf{E}$ is singular and a system of ordinary differential equations if $\mathbf{E}$ is nonsingular.

The assumption that $\mathbf{A} - s\,\mathbf{E}$ is regular is satisfied for all meaningful applications that lead to systems of the form (1). When in doubt, this assumption can be checked by means of the QZ algorithm; see, e.g., [16, Chapter 7.7]. The QZ algorithm generates unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that the matrices $\mathbf{Q}^{\mathrm{H}}\,\mathbf{A}\,\mathbf{Z} =: \mathbf{U} = \big[u_{jk}\big]$ and $\mathbf{Q}^{\mathrm{H}}\,\mathbf{E}\,\mathbf{Z} =: \mathbf{V} = \big[v_{jk}\big]$ are both upper triangular. The pencil $\mathbf{A} - s\,\mathbf{E}$ is regular if, and only if, for all $k$, $u_{kk}$ and $v_{kk}$ are not both zero.

The input-output behavior of the system (1) is described by the *transfer function*

$$g(s) := \mathbf{c}^{\mathrm{T}}\,(s\,\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{b} + d. \tag{2}$$

More precisely, assuming zero initial conditions, $Y(s) = g(s)U(s)$, where $U(s)$ and $Y(s)$ denote the Laplace transforms of the input $u(t)$ and output $y(t)$ of (1), respectively. The function (2) is rational with numerator and denominator degrees at most $n$. We remark that we do not assume *minimality* of the state-variable description (1). This means that, for a given transfer function $g$, the size $n$ of the matrices $\mathbf{A}, \mathbf{E}$ and of the vectors $\mathbf{b}$, $\mathbf{c}$ in the representation (2) of $g$ is not necessarily minimal.

An important special case is systems (1) with $\mathbf{E} = \mathbf{I}$. In this case, the pencil $\mathbf{A} - s\,\mathbf{I}$ reduces to a standard eigenvalue problem. As a result, the eigenvalue-based tests presented in this paper simplify considerably if $\mathbf{E} = \mathbf{I}$, and therefore, we will discuss this special case separately. To distinguish the general and this special case, we refer to $g$ given by (2) as a *general transfer function*, and for $\mathbf{E} = \mathbf{I}$, we refer to

$$h(s) := \mathbf{c}^{\mathrm{T}}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} + d \tag{3}$$

as a *standard transfer function.*

The following well-known theorem (see, e.g., [3]) relates the passivity of the system (1) to the positive realness of its transfer function.

**Theorem A** *The system* (1) *is passive if, and only if, the associated transfer function* (2) *is positive real.*

The definition of a positive real function is as follows; see, e.g., [3].

**Definition 1** *A function* $f : \mathbb{C} \mapsto \mathbb{C} \cup \{\infty\}$ *is called positive real if*

(i) $f$ *is analytic in* $\mathbb{C}_+$*;*

(ii) $f(\bar{s}) = \overline{f(s)}$ *for all* $s \in \mathbb{C}$*;*

(iii) $\mathrm{Re}(f(s)) \geq 0$ *for all* $s \in \mathbb{C}_+$.

Using standard results from complex analysis, such as the Maximum Modulus Theorem, one readily obtains the following well-known conditions for a rational function to be positive real; see, e.g., [17, Chapter 1] or [28, Chapter 4].

**Theorem B** *Let* $f$ *be a rational function.*

a) *If* $f$ *is positive real, then it has no poles and zeros in* $\mathbb{C}_+$ *and any possible pole and zero on the imaginary axis* (*including* $s = \infty$) *is simple.*

b) *The function* $f$ *is positive real if, and and if, it has no poles in* $\mathbb{C}_+$ *and*

$$\mathrm{Re}\left(f(\mathtt{i}\,\omega)\right) \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{4}$$

The procedures developed in this paper test the positive realness of transfer functions by checking the conditions given in part b) of Theorem B via the solution of standard or generalized eigenvalue problems.

An important special case is positive real functions for which (4) holds with equality. More precisely, a positive real function $f$ is called *lossless positive real* if $\mathrm{Re}(f(\mathtt{i}\,\omega)) = 0$ for all $\omega \in \mathbb{R}$ for which $\mathtt{i}\,\omega$ is not a pole of $f$. It can be shown that all poles and zeros of a lossless positive real function are purely imaginary; see, e.g., [3].

# 3 Eigenvalue-Based Characterization

In this section, we derive our eigenvalue-based characterization of positive realness.

## 3.1 Assumptions

From now on, we always assume that the $n \times n$ matrix pencil $\mathbf{A} - s\,\mathbf{E}$ is regular. We remark that if the transfer function $g$ in (2) is constant, i.e., $g(s) = \mathrm{const}$ for all $s$, then $g$ is positive real if, and only if, $\mathrm{const} \geq 0$. Therefore, from now on, we exclude this trivial case and assume that the transfer function $g$ is not the constant function. Note that, if in doubt, it is easy to check if $g$ is constant. Indeed, the representation (2) implies that $g$ is a rational function with numerator and denominator degrees at most $n$. Such a function can attain the same value at at most $n$ different points $s_j$, unless the function is constant. Therefore, $g$ is constant if, and only if, for any set of $n+1$ distinct points $s_j$, $j = 1, 2, \ldots, n+1$, the function values $g(s_j)$, $j = 1, 2, \ldots, n+1$, are identical. Checking if $g$ is constant thus can be done by evaluating $g$ at $n+1$ distinct points.

### 3.2 Reduction to Unit Vectors

The first step is a suitable state-space transformation to simplify the transfer function. This is done by means of the following lemma on the reduction of a pair of vectors to unit vectors.

**Lemma 1** *Let* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, *and assume that* $\mathbf{w}^{\mathrm{T}}\mathbf{v} \neq 0$. *Then, there exists a nonsingular matrix* $\mathbf{Q} \in \mathbb{R}^{n \times n}$ *and scalars* $\gamma_v, \gamma_w \in \mathbb{R}$ *such that*

$$\mathbf{Q}\,\mathbf{v} = \gamma_v\,\mathbf{e}_1, \quad \mathbf{Q}^{-\mathrm{T}}\,\mathbf{w} = \gamma_w\,\mathbf{e}_1, \quad \textit{and} \quad \mathbf{w}^{\mathrm{T}}\mathbf{v} = \gamma_v\,\gamma_w. \tag{5}$$

*Here,* $\mathbf{e}_1$ *denotes the first unit vector of length* $n$.

**Proof.** We give a constructive proof for the existence of a matrix $\mathbf{Q}$ satisfying (5). In particular, the proof itself provides a numerical method for computing $\mathbf{Q}$.

First, using an orthogonal Householder matrix $\mathbf{Q}_1 \in \mathbb{R}^{n \times n}$ (see, e.g., [16, Chapter 5.1]), we transform $\mathbf{v}$ into a multiple of the first unit vector:

$$\mathbf{Q}_1\,\mathbf{v} = \gamma_v\,\mathbf{e}_1, \quad \text{where} \quad \gamma_v \in \mathbb{R}. \tag{6}$$

Next, using an orthogonal Householder matrix $\mathbf{Q}_2 \in \mathbb{R}^{(n-1)\times(n-1)}$, we zero out the trailing $n-2$ entries of the vector $\mathbf{v} := \mathbf{Q}_1\,\mathbf{w}$:

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} \mathbf{Q}_1\,\mathbf{w} = \begin{bmatrix} \gamma_w \\ \eta \\ \mathbf{0} \end{bmatrix}, \quad \text{where} \quad \gamma_w, \eta \in \mathbb{R}. \tag{7}$$

Note that, by (6), we have

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} \mathbf{Q}_1\,\mathbf{v} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} (\gamma_v\,\mathbf{e}_1) = \gamma_v\,\mathbf{e}_1. \tag{8}$$

Taking inner products of the vectors in (7) and (8), we get $\gamma_v\,\gamma_w = \mathbf{w}^{\mathrm{T}}\,\mathbf{v} \neq 0$, which is just the third equation in (5). In particular, $\gamma_w \neq 0$, and we set

$$\mathbf{R} := \begin{bmatrix} 1 & \eta/\gamma_w \\ 0 & 1 \end{bmatrix}.$$

The matrix $\mathbf{R}$ satisfies the relations

$$\mathbf{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{R}^{-\mathrm{T}} \begin{bmatrix} \gamma_w \\ \eta \end{bmatrix} = \begin{bmatrix} \gamma_w \\ 0 \end{bmatrix}. \tag{9}$$

Finally, we set

$$\mathbf{Q} := \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-2} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix} \mathbf{Q}_1.$$

Using (7)–(9), one readily verifies that $\mathbf{Q}$ indeed satisfies the first two relations in (5). ■

### 3.3 Standard Transfer Functions

In this subsection, we consider the special case of standard transfer functions (3). Without loss of generality, we assume $d = 0$ in (3) so that $h$ is of the form

$$h(s) = \mathbf{c}^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{b}. \tag{10}$$

Indeed, if $d \neq 0$ in (3), we recast $h$ as

$$h(s) = \begin{bmatrix} \mathbf{c}^{\mathrm{T}} & 1 \end{bmatrix} \left( s \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & -d^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{b} \\ 1 \end{bmatrix}$$

and treat it as a general transfer function using the techniques discussed in Section 3.4 below.

**Proposition 1** *Let $h$ be a non-constant function of the form* (10). *If $h$ is positive real, then* $\mathbf{c}^{\mathrm{T}}\mathbf{b} > 0$.

**Proof.** Expanding the function $h$ in (10) about $s = \infty$, we get

$$h(s) = \frac{1}{s}\,\mathbf{c}^{\mathrm{T}} \left( \mathbf{I} - \frac{1}{s}\,\mathbf{A} \right)^{-1} \mathbf{b} = \sum_{j=0}^{\infty} \frac{\mathbf{c}^{\mathrm{T}}\,\mathbf{A}^{j}\,\mathbf{b}}{s^{j+1}}. \tag{11}$$

Now assume that $h$ is positive real. By parts (ii) and (iii) of Definition 1, $h(s) \geq 0$ for all $s > 0$, and thus $s h(s) \geq 0$ for all $s > 0$. Together with (11), it follows that $\lim_{s \to \infty} s h(s) = \mathbf{c}^{\mathrm{T}}\mathbf{b} \geq 0$. Now suppose that $\mathbf{c}^{\mathrm{T}}\mathbf{b} = 0$. Since $h$ is assumed to be non-constant, not all the coefficients in (11) can be zero. Let $k \geq 1$ be the smallest integer such that $\mathbf{c}^{\mathrm{T}}\,\mathbf{A}^{k}\,\mathbf{b} \neq 0$. Then, by (11),

$$h(s) = \sum_{j=k}^{\infty} \frac{\mathbf{c}^{\mathrm{T}}\,\mathbf{A}^{j}\,\mathbf{b}}{s^{j+1}} = \frac{\mathbf{c}^{\mathrm{T}}\,\mathbf{A}^{k}\,\mathbf{b}}{s^{k+1}} \left( 1 + \mathcal{O}\left(\frac{1}{s}\right) \right), \quad \text{where} \quad \mathbf{c}^{\mathrm{T}}\,\mathbf{A}^{k}\,\mathbf{b} \neq 0.$$

This means that $s = \infty$ is a zero of $h$ of multiplicity $k + 1 \geq 2$. However, by part a) of Theorem B, this is impossible. Thus $\mathbf{c}^{\mathrm{T}}\mathbf{b} > 0$. ∎

In view of Proposition 1, from now on, we assume that $\mathbf{c}^{\mathrm{T}}\mathbf{b} > 0$. By Lemma 1 (applied to the vectors $\mathbf{v} = \mathbf{b}$ and $\mathbf{w} = \mathbf{c}$), there exists a nonsingular matrix $\mathbf{Q}$ such that

$$\mathbf{Q}\,\mathbf{b} = \gamma_b\,\mathbf{e}_1, \quad \mathbf{Q}^{-\mathrm{T}}\,\mathbf{c} = \gamma_c\,\mathbf{e}_1, \quad \text{and} \quad \mathbf{c}^{\mathrm{T}}\mathbf{b} = \gamma_b\,\gamma_c. \tag{12}$$

By means of the state-space transformation $\mathbf{x} \mapsto \mathbf{Q}\,\mathbf{x}$ and by using (12), the transfer function (10) can be rewritten as follows:

$$h(s) = \mathbf{c}^{\mathrm{T}}\mathbf{b}\,\mathbf{e}_1^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{T}\right)^{-1} \mathbf{e}_1, \quad \text{where} \quad \mathbf{T} := \mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^{-1}. \tag{13}$$

Recall that $\mathbf{e}_1$ is the first unit vector. Using Cramer's rule, it follows that

$$\mathbf{e}_1^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{T}\right)^{-1} \mathbf{e}_1 = \frac{\det(s\,\mathbf{I} - \widehat{\mathbf{T}})}{\det(s\,\mathbf{I} - \mathbf{T})}. \tag{14}$$

Here and in the sequel, $\widehat{\mathbf{T}}$ denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the first column of $\mathbf{T}$. Inserting (14) into (13), we get

$$h(s) = \mathbf{c}^{\mathrm{T}}\mathbf{b}\,\frac{\det(s\,\mathbf{I} - \widehat{\mathbf{T}})}{\det(s\,\mathbf{I} - \mathbf{T})}. \tag{15}$$

From (15), we immediately obtain the following characterization of the poles and zeros of the transfer function $h$ in terms of the non-common eigenvalues of the matrices $\mathbf{T}$ and $\widehat{\mathbf{T}}$.

**Lemma 2** *Let* $h(s) = \mathbf{c}^{\mathrm{T}}\mathbf{b}\, \mathbf{e}_1^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{T}\right)^{-1} \mathbf{e}_1$. *Then:*

a) *The poles of* $h$ *are the eigenvalues of* $\mathbf{T}$ *that are not common with* $\widehat{\mathbf{T}}$*;*

b) *The zeros of* $h$ *are the eigenvalues of* $\widehat{\mathbf{T}}$ *that are not common with* $\mathbf{T}$*.*

*In particular, if* $\mathbf{T}$ *and* $\widehat{\mathbf{T}}$ *have no common eigenvalues, then the poles of* $h$ *are the eigenvalues of* $\mathbf{T}$ *and the zeros of* $h$ *are the eigenvalues of* $\widehat{\mathbf{T}}$*.*

Note that, by (13), the matrices $\mathbf{T}$ and $\mathbf{A}$ are similar, and thus $\mathbf{T}$ and $\mathbf{A}$ have the same eigenvalues.

By part b) of Theorem B, the transfer function $h$, (10), is positive real if, and only if, none of its poles lie in $\mathbb{C}_+$ and $h$ satisfies (4). In view of Lemma 2, the condition on the poles of $h$ can be checked by means of eigenvalue computations. Next, we show how condition (4) can also be checked via an eigenvalue problem derived from the representation (13) of $h$.

Let $\omega \in \mathbb{R}$. Using (13), we get

$$\begin{aligned} 2\,\mathrm{Re}\left(h(\mathrm{i}\,\omega)\right) &= h(\mathrm{i}\,\omega) + h(-\mathrm{i}\,\omega) \\ &= \mathbf{c}^{\mathrm{T}}\mathbf{b}\,\mathbf{e}_1^{\mathrm{T}} \left( \left(\mathrm{i}\,\omega\,\mathbf{I} - \mathbf{T}\right)^{-1} + \left(-\mathrm{i}\,\omega\,\mathbf{I} - \mathbf{T}\right)^{-1} \right) \mathbf{e}_1 \\ &= -2\,\mathbf{c}^{\mathrm{T}}\mathbf{b}\,\mathbf{e}_1^{\mathrm{T}} \left(\mathrm{i}\,\omega\,\mathbf{I} - \mathbf{T}\right)^{-1} \left(-\mathrm{i}\,\omega\,\mathbf{I} - \mathbf{T}\right)^{-1} \mathbf{T}\,\mathbf{e}_1 \\ &= -2\,\mathbf{c}^{\mathrm{T}}\mathbf{b}\,\mathbf{e}_1^{\mathrm{T}} \left(\mathbf{T}^2 + \omega^2\,\mathbf{I}\right)^{-1} \mathbf{T}\,\mathbf{e}_1 = -2\,\mathbf{c}^{\mathrm{T}}\mathbf{b}\,\mathbf{e}_1^{\mathrm{T}}\,\mathbf{z}(\omega), \end{aligned} \tag{16}$$

where $\mathbf{z}(\omega)$ denotes the solution of the system of linear equations

$$\left(\mathbf{T}^2 + \omega^2\,\mathbf{I}\right)\,\mathbf{z}(\omega) = \mathbf{T}\,\mathbf{e}_1. \tag{17}$$

Using Cramer's rule, we deduce from (17) that

$$\mathbf{e}_1^{\mathrm{T}}\,\mathbf{z}(\omega) = \frac{\det(\mathbf{M} + \omega^2\,\mathbf{N})}{\det(\mathbf{T}^2 + \omega^2\,\mathbf{I})}, \tag{18}$$

where $\mathbf{M} + \omega^2\,\mathbf{N}$ denotes the matrix pencil obtained by replacing the first column of the matrix pencil $\mathbf{T}^2 + \omega^2\,\mathbf{I}$ by the vector $\mathbf{T}\,\mathbf{e}_1$. It is easy to verify that $\mathbf{M}$ and $\mathbf{N}$ are given by

$$\mathbf{M} = \mathbf{T}^2\left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right) + \mathbf{T}\,\mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}} \quad \text{and} \quad \mathbf{N} = \mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}. \tag{19}$$

Next, note that, since $\mathbf{T}$ is real, we have

$$\begin{aligned} \det\left(\mathbf{T}^2 + \omega^2\,\mathbf{I}^2\right) &= \det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right) \cdot \left(\mathbf{T} - \mathrm{i}\,\omega\,\mathbf{I}\right) \\ &= \det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right) \cdot \det\left(\mathbf{T} - \mathrm{i}\,\omega\,\mathbf{I}\right) \\ &= \det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right) \cdot \overline{\det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right)} \\ &= \left|\det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right)\right|^2 \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R}. \end{aligned} \tag{20}$$

By (18) and (20), it follows from (16) that

$$\mathrm{Re}\left(h(\mathrm{i}\,\omega)\right) = -\mathbf{c}^{\mathrm{T}}\mathbf{b}\,\frac{\det\left(\mathbf{M} + \omega^2\,\mathbf{N}\right)}{\left|\det\left(\mathbf{T} + \mathrm{i}\,\omega\,\mathbf{I}\right)\right|^2} \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{21}$$

Recall that we assumed $\mathbf{c}^{\mathrm{T}}\mathbf{b} > 0$. Moreover, by (20), the denominator in (21) is non-negative. Therefore, by the representation (21) of $\mathrm{Re}\,(h(\mathtt{i}\,\omega))$, the condition (4) for positive realness in Theorem B is satisfied if, and only if,

$$p(\lambda) := \det\left(\mathbf{M} + \lambda\,\mathbf{N}\right) \leq 0 \quad \text{for all} \quad \lambda \geq 0. \tag{22}$$

Here, we have set $\lambda := \omega^2$.

The condition (22) is trivially satisfied if the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ is singular. In this case, $p = 0$ is the zero polynomial, and thus, by (21), $\mathrm{Re}\,(h(\mathtt{i}\,\omega)) = 0$ for all $\omega \in \mathbb{R}$ for which $\mathtt{i}\,\omega$ is not a pole. Recall that this is just the condition for $h$ to be lossless positive real.

Now assume that the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ is regular. In this case, $p$ is a real polynomial of degree $n-1$. Note that $p(\lambda)$ can change its sign at a real $\lambda$ only if $\lambda$ is a zero of $p$ and if its multiplicity is odd. Therefore, the condition (22) is satisfied if, and only if, $p$ has no positive zeros of odd multiplicity and $p(\lambda_0) < 0$ for at least one point $\lambda_0 > 0$ that is not a zero of $p$. By (22), the zeros of $p$ are just the finite eigenvalues of the matrix pencil $\mathbf{M} + \lambda\,\mathbf{N}$.

In summary, we have obtained the following theorem, which gives an eigenvalue-based characterization of positive realness of transfer functions of the form (10).

**Theorem 1** *Let* $h(s) = \mathbf{c}^{\mathrm{T}}\left(s\,\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{b}$ *be a given transfer function, and assume that* $h$ *is not constant. Let* $\mathbf{T}$ *be the matrix defined by the transformation* (13) *of* $h$, *and let* $\mathbf{M}$ *and* $\mathbf{N}$ *be the matrices given by* (19). *Then,* $h$ *is positive real if, and only if, the following conditions are satisfied* :

(i) $\mathbf{c}^{\mathrm{T}}\mathbf{b} > 0$;

(ii) *None of the eigenvalues of* $\mathbf{T}$ *that are not common with* $\widehat{\mathbf{T}}$ *lie in* $\mathbb{C}_+$;

(iii) *The matrix pencil* $\mathbf{M} + \lambda\,\mathbf{N}$ *is either singular, or it is regular and*

$\alpha$) *Any possible real positive eigenvalue,* $\lambda > 0$, *of* $\mathbf{M} + \lambda\,\mathbf{N}$ *has even multiplicity;*

$\beta$) $\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right) < 0$ *for at least one point* $\lambda_0 > 0$.

*Furthermore, if* $\mathbf{M} + \lambda\,\mathbf{N}$ *is singular, then* $h$ *is lossless positive real.*

### 3.4 General Transfer Functions

In this subsection, we consider the case of general transfer functions (2). Again, without loss of generality, we assume $d = 0$ in (2) so that $g$ is of the form

$$g(s) = \mathbf{c}^{\mathrm{T}}\left(s\,\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{b}. \tag{23}$$

Indeed, if $d \neq 0$, we rewrite $g$ in the form (23) as follows:

$$g(s) = \begin{bmatrix}\mathbf{c}^{\mathrm{T}} & 1\end{bmatrix}\left(s\begin{bmatrix}\mathbf{E} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix} - \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & -d^{-1}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{b}\\ 1\end{bmatrix}.$$

This means that the case $d \neq 0$ can be treated as a general transfer function of the form (23) with state-space dimension $n+1$, instead of $n$.

First, we rewrite $g$ in terms of one matrix, $\mathbf{G}$, instead of $\mathbf{E}$ and $\mathbf{A}$. To this end, we choose an expansion point $s_0 \in \mathbb{R}$ such that

$$g(s_0) \neq 0 \quad \text{and the matrix } \mathbf{A} - s_0\,\mathbf{E} \text{ is nonsingular.} \tag{24}$$

Our assumptions that the function $g$ is not constant and that the matrix pencil $\mathbf{A} - s\,\mathbf{E}$ is regular guarantee the existence of points $s_0$ with (24). Let

$$\mathbf{A} - s_0\,\mathbf{E} = \mathbf{L}\,\mathbf{U}, \quad \text{where} \quad \mathbf{L},\, \mathbf{U} \in \mathbb{R}^{n\times n}, \tag{25}$$

be any formal factorization of $\mathbf{A} - s_0\,\mathbf{E}$. For example, (25) may be the result of a true LU factorization of $\mathbf{A} - s_0\,\mathbf{E}$ into a lower and upper triangular matrix $\mathbf{L}$ and $\mathbf{U}$, or (25) may be the trivial factorization with $\mathbf{L} := \mathbf{A} - s_0\,\mathbf{E}$ and $\mathbf{U} := \mathbf{I}$. Using (25), we then rewrite the transfer function (23) as follows:

$$\begin{aligned} g(s) &= \mathbf{l}^{\mathrm{T}}\,\big(\mathbf{I} - (s - s_0)\,\mathbf{G}\big)^{-1}\,\mathbf{r}, \\ \text{where} \quad \mathbf{G} &:= \mathbf{L}^{-1}\,\mathbf{E}\,\mathbf{U}^{-1}, \quad \mathbf{r} := -\mathbf{L}^{-1}\,\mathbf{b}, \quad \mathbf{l} := \mathbf{U}^{-\mathrm{T}}\,\mathbf{c}. \end{aligned} \tag{26}$$

By (26) and (24), we have $\mathbf{l}^{\mathrm{T}}\mathbf{r} = g(s_0) \neq 0$. Therefore, we can use Lemma 1 (applied to the vectors $\mathbf{v} = \mathbf{r}$ and $\mathbf{w} = \mathbf{l}$) to construct a nonsingular matrix $\mathbf{Q}$ such that

$$\mathbf{Q}\,\mathbf{r} = \gamma_r\,\mathbf{e}_1, \quad \mathbf{Q}^{-\mathrm{T}}\,\mathbf{l} = \gamma_l\,\mathbf{e}_1, \quad \text{and} \quad \mathbf{l}^{\mathrm{T}}\mathbf{r} = \gamma_r\,\gamma_l. \tag{27}$$

Using the state-space transformation $\mathbf{x} \mapsto \mathbf{Q}\,\mathbf{x}$ and the relations in (27), the representation (26) of $g$ can be rewritten as follows:

$$\begin{aligned} g(s) &= \mathbf{l}^{\mathrm{T}}\mathbf{r}\;\mathbf{e}_1^{\mathrm{T}}\,(\mathbf{S} - s\,\mathbf{T})^{-1}\,\mathbf{e}_1, \\ \text{where} \quad \mathbf{T} &:= \mathbf{Q}\,\mathbf{G}\,\mathbf{Q}^{-1} \quad \text{and} \quad \mathbf{S} := \mathbf{I} + s_0\,\mathbf{T}. \end{aligned} \tag{28}$$

By Cramer's rule, it follows from (28) that

$$g(s) = \mathbf{l}^{\mathrm{T}}\mathbf{r}\;\frac{\det\big(\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}\big)}{\det\,(\mathbf{S} - s\,\mathbf{T})}, \tag{29}$$

where $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$ is the pencil obtained by deleting the first row and the first column of the pencil $\mathbf{S} - s\,\mathbf{T}$. The representation (29) immediately shows the following characterization of the poles and zeros of $g$.

**Lemma 3** *Let* $g(s) = \mathbf{l}^{\mathrm{T}}\mathbf{r}\;\mathbf{e}_1^{\mathrm{T}}\,(\mathbf{S} - s\,\mathbf{T})^{-1}\,\mathbf{e}_1$. *Then :*

a) *The poles of* $g$ *are the eigenvalues of the pencil* $\mathbf{S} - s\,\mathbf{T}$ *that are not common with* $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$*;*

b) *The zeros of* $g$ *are the eigenvalues of the pencil* $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$ *that are not common with* $\mathbf{S} - s\,\mathbf{T}$.

*In particular, if* $\mathbf{S} - s\,\mathbf{T}$ *and* $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$ *have no common eigenvalues, then the poles of* $g$ *are the eigenvalues of* $\mathbf{S} - s\,\mathbf{T}$ *and the zeros of* $g$ *are the eigenvalues of* $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$.

We remark that the pencils $\mathbf{S} - s\,\mathbf{T}$ and $\mathbf{A} - s\,\mathbf{E}$ are equivalent and thus have the same eigenvalues. Indeed, using (25), (26), and (28), one readily verifies that

$$\mathbf{S} - s\,\mathbf{T} = \left(\mathbf{Q}\,\mathbf{L}^{-1}\right)(\mathbf{A} - s\,\mathbf{E})\left(\mathbf{U}^{-1}\,\mathbf{Q}^{-1}\right).$$

By part b) of Theorem B, the transfer function $g$, (23), is positive real if, and only if, none of its poles lie in $\mathbb{C}_+$ and $g$ satisfies (4). In view of Lemma 3, the condition on the poles of $g$ can be checked by means of eigenvalue computations. Next, we show how condition (4) can also be checked via an eigenvalue problem derived from the representation (28) of $g$.

Let $\omega \in \mathbb{R}$. Using (28), we get

$$\begin{aligned} 2\,\mathrm{Re}(g(\mathtt{i}\,\omega)) &= g(\mathtt{i}\,\omega) + g(-\mathtt{i}\,\omega) \\ &= \mathbf{l}^{\mathrm{T}}\mathbf{r}\,\mathbf{e}_1^{\mathrm{T}}\left((\mathbf{S} - \mathtt{i}\,\omega\,\mathbf{T})^{-1} + (\mathbf{S} + \mathtt{i}\,\omega\,\mathbf{T})^{-1}\right)\mathbf{e}_1 \\ &= 2\,\mathbf{l}^{\mathrm{T}}\mathbf{r}\,\mathbf{e}_1^{\mathrm{T}}\,(\mathbf{S} - \mathtt{i}\,\omega\,\mathbf{T})^{-1}\,(\mathbf{S} + \mathtt{i}\,\omega\mathbf{T})^{-1}\,\mathbf{S}\,\mathbf{e}_1 \\ &= 2\,\mathbf{l}^{\mathrm{T}}\mathbf{r}\,\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{S}^2 + \omega^2\,\mathbf{T}^2\right)^{-1}\mathbf{S}\,\mathbf{e}_1 = 2\,\mathbf{l}^{\mathrm{T}}\mathbf{r}\,\mathbf{e}_1^{\mathrm{T}}\,\mathbf{z}(\omega), \end{aligned} \tag{30}$$

where $\mathbf{z}(\omega)$ denotes the solution of the system of linear equations

$$\left(\mathbf{S}^2 + \omega^2\,\mathbf{T}^2\right)\,\mathbf{z}(\omega) = \mathbf{S}\,\mathbf{e}_1. \tag{31}$$

Using Cramer's rule, we deduce from (31) that

$$\mathbf{e}_1^{\mathrm{T}}\,\mathbf{z}(\omega) = \frac{\det\left(\mathbf{M} + \omega^2\,\mathbf{N}\right)}{\det\left(\mathbf{S}^2 + \omega^2\,\mathbf{T}^2\right)}, \tag{32}$$

where $\mathbf{M} + \omega^2\,\mathbf{N}$ is the matrix pencil given by

$$\mathbf{M} = \mathbf{S}^2\left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right) + \mathbf{S}\,\mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}} \quad \text{and} \quad \mathbf{N} = \mathbf{T}^2\left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right). \tag{33}$$

*Remark 1.* If $g(0) \neq 0$ and the matrix $\mathbf{A}$ is nonsingular, then we can use the trivial "shift" $s_0 = 0$ in (24). In this case, by (28), $\mathbf{S} = \mathbf{I}$, and thus the matrix $\mathbf{M}$ in (33) reduces to $\mathbf{M} = \mathbf{I}$.

Since $\mathbf{S}$ and $\mathbf{T}$ are real, we have

$$\det\left(\mathbf{S}^2 + \omega^2\,\mathbf{T}^2\right) = \left|\det\left(\mathbf{S} + \mathtt{i}\,\omega\,\mathbf{T}\right)\right|^2 \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{34}$$

Inserting (34) into (32) and the result into (30), we get

$$\mathrm{Re}(g(\mathtt{i}\,\omega)) = \mathbf{l}^{\mathrm{T}}\mathbf{r}\,\frac{\det\left(\mathbf{M} + \omega^2\,\mathbf{N}\right)}{\left|\det\left(\mathbf{S} + \mathtt{i}\,\omega\,\mathbf{T}\right)\right|^2} \quad \text{for all} \quad \omega \in \mathbb{R}, \tag{35}$$

where the denominator in (35) is always non-negative. Therefore, it follows from the representation (35) of $\mathrm{Re}(g(\mathtt{i}\,\omega))$ that $g$ satisfies condition (4) if, and only if,

$$\mathbf{l}^{\mathrm{T}}\mathbf{r}\,q(\lambda) \geq 0 \quad \text{for all} \quad \lambda \geq 0. \tag{36}$$

Here, we have set

$$\lambda := \omega^2 \quad \text{and} \quad q(\lambda) := \det\left(\mathbf{M} + \lambda\,\mathbf{N}\right). \tag{37}$$

If the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ is singular, then $q = 0$ is the zero polynomial and (36) is trivially satisfied. In this case, the transfer function $g$ is lossless positive real.

Now assume that $\mathbf{M} + \lambda\,\mathbf{N}$ is regular. In this case, $q$ is a real polynomial of degree at most $n-1$. Moreover, recall that, by (26) and (24), $\mathbf{l}^{\mathrm{T}}\mathbf{r} \neq 0$. Therefore, in (36), $\mathbf{l}^{\mathrm{T}}\mathbf{r}\; q(\lambda)$ can change its sign at a real $\lambda$ only if $\lambda$ is a zero of $q$ and if its multiplicity is odd. By (37), the zeros of $q$ are just the finite eigenvalues of the matrix pencil $\mathbf{M} + \lambda\,\mathbf{N}$.

In summary, we have obtained the following theorem, which gives an eigenvalue-based characterization of positive realness of general transfer functions of the form (23).

**Theorem 2** *Let $g(s) = \mathbf{c}^{\mathrm{T}}\left(s\,\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{b}$ be a given transfer function, and assume that $g$ is not constant. Let $s_0 \in \mathbb{R}$ be any point satisfying* (24). *Let $\mathbf{T}$ and $\mathbf{S}$ be the matrices defined by the transformation* (28) *of $g$, and let $\mathbf{M}$ and $\mathbf{N}$ be the matrices given by* (33). *Then, $g$ is positive real if, and only if, the following conditions are satisfied :*

(i) *None of the finite eigenvalues of the pencil $\mathbf{S} - s\,\mathbf{T}$ that are not common with $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$ lie in $\mathbb{C}_+$;*

(ii) *The matrix pencil $\mathbf{M} + \lambda\,\mathbf{N}$ is either singular, or it is regular and*

  $\alpha$) *Any possible real positive eigenvalue, $\lambda > 0$, of $\mathbf{M} + \lambda\,\mathbf{N}$ has even multiplicity;*

  $\beta$) $\mathbf{l}^{\mathrm{T}}\mathbf{r}\;\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right) > 0$ *for at least one point $\lambda_0 > 0$.*

*Furthermore, if $\mathbf{M} + \lambda\,\mathbf{N}$ is singular, then $g$ is lossless positive real.*

# 4 Eigenvalue-Based Tests

In this section, we present two computational procedures for testing positive realness. These tests are based on the results stated in Theorems 1 and 2. As in Section 3, we again treat the cases of standard and general transfer functions separately.

## 4.1 Standard Transfer Functions

Given a non-constant function $h$ of the form (10), the following algorithm determines if $h$ is positive real.

**Algorithm 1** (Test of positive realness of standard transfer functions)
INPUT: *The data $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{c}, \mathbf{b} \in \mathbb{R}^n$ of a non-constant function $h(s) = \mathbf{c}^{\mathrm{T}}\left(s\,\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{b}$.*

1. *Compute $\gamma = \mathbf{c}^{\mathrm{T}}\mathbf{b}$. If $\gamma \leq 0$, stop : $h$ is not positive real.*
2. *Use Lemma 1 to compute a matrix $\mathbf{Q}$ such that $\mathbf{Q}\,\mathbf{b} = \gamma_b\,\mathbf{e}_1$ and $\mathbf{Q}^{-\mathrm{T}}\,\mathbf{c} = \gamma_c\,\mathbf{e}_1$.*
3. *Set $\mathbf{T} = \mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^{-1}$.*
4. *Compute the eigenvalues of $\mathbf{T}$. If $\mathbf{T}$ has no eigenvalues with positive real part, go to Step 5. Otherwise, check if all eigenvalues of $\mathbf{T}$ with positive real part are also eigenvalues of $\widehat{\mathbf{T}}$, which is the matrix obtained by deleting the first row and the first column of $\mathbf{T}$. If no, stop : $h$ is not positive real.*
5. *Set $\mathbf{M} = \mathbf{T}^2\left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right) + \mathbf{T}\,\mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}$ and $\mathbf{N} = \mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}$.*

6. *If the pencil* $\mathbf{M} + \lambda\,\mathbf{N}$ *is singular, then stop : h is lossless positive real.*

7. *Compute the eigenvalues of the pencil* $\mathbf{M} + \lambda\,\mathbf{N}$. *If there is any real positive eigenvalue with odd multiplicity, stop : h is not positive real.*

8. *Select any* $\lambda_0 > 0$ *that is not an eigenvalue of* $\mathbf{M} + \lambda\,\mathbf{N}$. *If* $\det(\mathbf{M} + \lambda_0\,\mathbf{N}) > 0$, *then h is not positive real. Otherwise, h is positive real.*

Next, we discuss some implementation issues for Algorithm 1.

*Remark 2.* In Steps 2 and 3, the matrix $\mathbf{Q}$ and its inverse never need to be formed explicitly. Indeed, as shown in the proof of Lemma 1, $\mathbf{Q}$ is the product of two Householder transformations and an identity matrix with one additional nonzero entry in the (1,2) position. When forming $\mathbf{T}$, the pre- and post-multiplication of $\mathbf{A}$ by $\mathbf{Q}$ and $\mathbf{Q}^{-1}$, respectively, can thus be performed by applying a total of four Householder transformations in the usual way and by two additional SAXPYs; see, e.g.,, [16, Chapter 5.1].

*Remark 3.* If the transfer function $h$ is *stable*, then it has no poles with positive real part. Therefore, Step 4 can be omitted if $h$ is known to be stable.

*Remark 4.* When Algorithm 1 is used to test the positive realness of transfer functions $h$ generated by Lanczos-based reduced-order modeling, for example by the PVL method [14], then $h$ is already given in the form (13), where $\mathbf{T}$ is the tridiagonal Lanczos matrix. Hence Steps 2 and 3 are not necessary in this case.

*Remark 5.* The matrices $\mathbf{M}$ and $\mathbf{N}$ can always be partitioned as follows:

$$\mathbf{M} = \begin{bmatrix} m_{11} & \mathbf{m}_{12}^{\mathrm{T}} \\ \mathbf{m}_{21} & \mathbf{M}_{22} \end{bmatrix}, \quad \text{where} \quad m_{11} \in \mathbb{R}, \quad \text{and} \quad \mathbf{N} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-1} \end{bmatrix}.$$

If $m_{11} \neq 0$, then

$$\det\left(\mathbf{M} + \lambda\,\mathbf{N}\right) = m_{11}\ \det\left(\lambda\,\mathbf{I}_{n-1} - \widehat{\mathbf{M}}_{22}\right), \quad \text{where} \quad \widehat{\mathbf{M}}_{22} := \frac{1}{m_{11}}\mathbf{m}_{21}\,\mathbf{m}_{12}^{\mathrm{T}} - \mathbf{M}_{22}. \tag{38}$$

Thus, the finite eigenvalues of the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ are just the eigenvalues of the matrix $\widehat{\mathbf{M}}_{22}$. Moreover, by (38), $\operatorname{sign}\left(\det\left(\mathbf{M} + \lambda\,\mathbf{N}\right)\right) = \operatorname{sign}(m_{11})$ for all sufficiently large $\lambda > 0$. As a result, in Step 8, it is sufficient to check the sign of $m_{11}$, instead of the sign of $\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right)$.

*Remark 6.* In general, to determine the sign of $\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right)$, one first computes an LU factorization of the matrix $\mathbf{M} + \lambda_0\,\mathbf{N}$. The sign of $\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right)$ can then be determined from the sign of the diagonal entries of the factors $\mathbf{L}$ and $\mathbf{U}$.

*Remark 7.* Algorithm 1 involves the computation of all eigenvalues of the matrix $\mathbf{T}$ (by the QR algorithm) and of all eigenvalues of the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ (by the QZ algorithm). The costs for the QR and QZ algorithm are $10n^3 + \mathcal{O}(n^2)$ and $30n^3 + \mathcal{O}(n^2)$ operations, respectively; see, e.g., [16, Chapters 7.5.6 and 7.7.7]. The LU factorization of $\mathbf{M} + \lambda_0\,\mathbf{N}$ to obtain the sign of $\det\left(\mathbf{M} + \lambda_0\,\mathbf{N}\right)$ requires $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ operations. All remaining computations in Algorithm 1 involve $\mathcal{O}(n^2)$ operations. Therefore, the overall operation count for Algorithm 1 is approximately $41n^3 + \mathcal{O}(n^2)$. The computational costs can be approximately halved if the generalized eigenvalue problem $\mathbf{M} + \lambda\,\mathbf{N}$ can be treated as a standard eigenvalue problem by using the approach discussed in Remark 5.

### 4.2 General Transfer Functions

Given a non-constant function $g$ of the form (23), the following algorithm determines if $g$ is positive real.

**Algorithm 2** (Test of positive realness of general transfer functions)

INPUT: *The data* $\mathbf{A}$, $\mathbf{E} \in \mathbb{R}^{n \times n}$, $\mathbf{c}$, $\mathbf{b} \in \mathbb{R}^{n}$ *of a non-constant function* $g(s) = \mathbf{c}^{\mathrm{T}} \left(s\,\mathbf{E} - \mathbf{A}\right)^{-1} \mathbf{b}$.

1. *Select a shift* $s_0 \in \mathbb{R}$ *such that* $\mathbf{A} - s_0\,\mathbf{E}$ *is nonsingular, and compute a formal factorization* $\mathbf{A} - s_0\,\mathbf{E} = \mathbf{L}\,\mathbf{U}$. *For example, a true LU factorization, or the trivial "factorization"* $\mathbf{L} := \mathbf{A} - s_0\,\mathbf{E}$ *and* $\mathbf{U} := \mathbf{I}$ *may be used.*
2. *Compute* $\mathbf{G} = \mathbf{L}^{-1}\,\mathbf{E}\,\mathbf{U}^{-1}$, $\mathbf{r} = -\mathbf{L}^{-1}\,\mathbf{b}$, *and* $\mathbf{l} = \mathbf{U}^{-\mathrm{T}}\mathbf{c}$.
3. *If* $\mathbf{r}^{\mathrm{T}}\mathbf{l} = 0$, *return to Step* 1 *and select a new shift* $s_0$.
4. *Use Lemma* 1 *to compute a matrix* $\mathbf{Q}$ *such that* $\mathbf{Q}\,\mathbf{r} = \gamma_r\,\mathbf{e}_1$ *and* $\mathbf{Q}^{-\mathrm{T}}\,\mathbf{l} = \gamma_l\,\mathbf{e}_1$.
5. *Set* $\mathbf{T} = \mathbf{Q}\,\mathbf{G}\,\mathbf{Q}^{-1}$ *and* $\mathbf{S} = \mathbf{I} + s_0\,\mathbf{T}$.
6. *Compute the eigenvalues of the pencil* $\mathbf{S} - s\,\mathbf{T}$. *If* $\mathbf{S} - s\,\mathbf{T}$ *has no eigenvalues with positive real part, go to Step* 7. *Otherwise, check if all eigenvalues of* $\mathbf{S} - s\,\mathbf{T}$ *with positive real part are also eigenvalues of the pencil* $\widehat{\mathbf{S}} - s\,\widehat{\mathbf{T}}$, *which is the pencil obtained by deleting the first row and the first column of* $\mathbf{S} - s\,\mathbf{T}$. *If no, stop* : $g$ *is not positive real.*
7. *Set* $\mathbf{M} = \mathbf{S}^2 \left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right) + \mathbf{S}\,\mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}$ *and* $\mathbf{N} = \mathbf{T}^2 \left(\mathbf{I} - \mathbf{e}_1\,\mathbf{e}_1^{\mathrm{T}}\right)$.
8. *If the pencil* $\mathbf{M} + \lambda\,\mathbf{N}$ *is singular, then stop* : $g$ *is lossless positive real.*
9. *Compute the eigenvalues of the pencil* $\mathbf{M} + \lambda\,\mathbf{N}$. *If there is any real positive eigenvalue with odd multiplicity, stop* : $g$ *is not positive real.*
10. *Select any* $\lambda_0 > 0$ *that is not an eigenvalue of* $\mathbf{M} + \lambda\,\mathbf{N}$. *If* $\mathbf{l}^{\mathrm{T}}\mathbf{r}\ \det(\mathbf{M} + \lambda_0\,\mathbf{N}) < 0$, *then* $g$ *is not positive real. Otherwise,* $g$ *is positive real.*

Next, we discuss some implementation issues for Algorithm 2.

*Remark 8.* Remark 2 about Algorithm 1 also applies to Steps 4 and 5 of Algorithm 2.

*Remark 9.* If $g$ is known to be stable, then Step 6 can be omitted.

*Remark 10.* For transfer functions of reduced-order models defined by Padé approximation, the main criterion for choosing the shift $s_0$ is the approximation quality of the reduced-order model; see, e.g., [14, 15]. When the shift $s_0$ is only used in the test of positive realness of $g$, then $s_0$ should be chosen such that $\mathbf{A} - s_0\,\mathbf{E}$ is a well-conditioned matrix.

*Remark 11.* When Algorithm 2 is used to test the positive realness of transfer functions $g$ generated by Lanczos-based reduced-order modeling, then $g$ is already given in the form (28), where $\mathbf{T}$ is the tridiagonal Lanczos matrix. Hence Steps 1–5 are not necessary in this case.

*Remark 12.* The first column of the matrix $\mathbf{N}$ is always zero. The matrices $\mathbf{M}$ and $\mathbf{N}$ can thus be partitioned conformally as follows:

$$\mathbf{M} = \begin{bmatrix} m_{11} & \mathbf{m}_{12}^{\mathrm{T}} \\ \mathbf{m}_{21} & \mathbf{M}_{22} \end{bmatrix}, \quad \text{where} \quad m_{11} \in \mathbb{R}, \quad \text{and} \quad \mathbf{N} = \begin{bmatrix} 0 & \mathbf{n}_{12}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{N}_{22} \end{bmatrix}.$$

By means of a Householder transformation, one can deflate the singularity due to the zero column of $\mathbf{N}$, before computing the eigenvalues of the pencil $\mathbf{M} + \lambda\,\mathbf{N}$. Specifically, let $\mathbf{H}$ be a Householder transformation such that

$$\mathbf{H} \begin{bmatrix} m_{11} \\ \mathbf{m}_{21} \end{bmatrix} = \widetilde{m}_{11}\,\mathbf{e}_1.$$

Then, by applying $\mathbf{H}$ to the pencil $\mathbf{M} + \lambda\,\mathbf{N}$, we get

$$\mathbf{H}\,(\mathbf{M} + \lambda\,\mathbf{N}) = \mathbf{H} \left( \begin{bmatrix} m_{11} & \mathbf{m}_{12}^{\mathrm{T}} \\ \mathbf{m}_{21} & \mathbf{M}_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & \mathbf{n}_{12}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{N}_{22} \end{bmatrix} \right) =: \begin{bmatrix} \widetilde{m}_{11} & \widetilde{\mathbf{m}}_{12}^{\mathrm{T}} \\ \mathbf{0} & \widetilde{\mathbf{M}}_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & \widetilde{\mathbf{n}}_{12}^{\mathrm{T}} \\ \mathbf{0} & \widetilde{\mathbf{N}}_{22} \end{bmatrix}.$$

If $\widetilde{m}_{11} = 0$, the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ is singular. Otherwise, $\lambda = -\widetilde{m}_{11}/0 = \infty$ is an infinite eigenvalue of $\mathbf{M} + \lambda\,\mathbf{N}$, and the remaining eigenvalues are just the eigenvalues of $\widetilde{\mathbf{M}}_{22} + \lambda\,\widetilde{\mathbf{N}}_{22}$.

*Remark 13.* In some applications, especially reduced-order modeling of VLSI circuits, the norms of the matrices $\mathbf{M}$ and $\mathbf{N}$ are often orders of magnitudes different. For improved numerical accuracy, a proper scaling should be used before computing the eigenvalues of $\mathbf{M} + \lambda\,\mathbf{N}$. More precisely, the re-scaled pencil $\mathbf{M} + (\lambda/\tau)\ (\tau\,\mathbf{N})$ should be used, where $\tau$ is chosen such that the norms of $\mathbf{M}$ and $\tau\,\mathbf{N}$ are of the same order of magnitude.

*Remark 14.* By using the operation counts for the basic matrix computations as stated in Remark 7 above, one concludes that the overall operation count for Algorithm 2 is approximately $62n^3 + \mathcal{O}(n^2)$.

## 5 Numerical Examples

In this section, we present seven examples to illustrate the use of Algorithms 1 and 2.

**Example 1.** In this example, we employ Algorithm 1 to test the positive realness of the standard transfer function

$$h(s) = \mathbf{c}^{\mathrm{T}}\,(s\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}. \tag{39}$$

Algorithm 1 first checks that $\mathbf{c}^{\mathrm{T}}\mathbf{b} = 1 > 0$ and then generates the transformed matrix

$$\mathbf{T} = \begin{bmatrix} -1.5 & -0.25 \\ -1 & -1.5 \end{bmatrix}. \tag{40}$$

The matrix $\mathbf{T}$ has the eigenvalues $-2$ and $-1$, and therefore, Algorithm 1 proceeds to generate from (40) the pencil

$$\mathbf{M} + \lambda\,\mathbf{N} = \begin{bmatrix} -1.5 & 0.75 \\ -1 & 2.5 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

whose only finite eigenvalue is $-2$. Moreover, $\det(\mathbf{M} + \lambda_0\,\mathbf{N}) = -4.5 < 0$ for $\lambda_0 = 1$. Therefore, Algorithm 1 found that the transfer function (39) is positive real.

Of course, for this simple example, positive realness can also be verified by direct computation. Indeed, from (39), we get

$$h(s) = \frac{s+1.5}{(s+1)(s+2)}, \tag{41}$$

which shows that $h$ has the poles $-1$ and $-2$. Furthermore, from (41), we obtain

$$\operatorname{Re}(h(\mathtt{i}\,\omega)) = \frac{1.5\,\omega^2+3}{(2-\omega^2)^2+9\,\omega^2} \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R},$$

and thus $h$ is positive real by Theorem B.

**Example 2.** This example shows that, in general, the necessary condition in part a) of Theorem B is not sufficient for positive realness. We consider the standard transfer function

$$h(s) = \mathbf{c}^{\mathrm{T}}\,(s\,\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \tag{42}$$

Algorithm 1 applied to (42) first produces the transformed matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}. \tag{43}$$

The only eigenvalue of $\mathbf{T}$ is $-1$ (with multiplicity 2), and so $h$ has no poles in $\mathbb{C}_+$. Algorithm 1 then proceeds to generate from (43) the pencil

$$\mathbf{M} + \lambda\,\mathbf{N} = \begin{bmatrix} 1 & -8 \\ -1 & 5 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This pencil has the real positive eigenvalue 3 with multiplicity 1, and therefore, the transfer function (42) is not positive real.

Note that, by Lemma 2 and (43), $\widehat{\mathbf{T}} = -3$ is the only zero of $h$. Thus, $h$ has no poles and zeros in $\mathbb{C}_+$, yet $h$ is not positive real. The reason is that the additional condition (4) in part b) of Theorem B is not satisfied. Indeed, from (42), by direct computation, one gets $h(s) = (s+3)/(s+1)^2$ and further

$$\operatorname{Re}(h(\mathtt{i}\,\omega)) = \frac{3-\omega^2}{(1+\omega^2)^2} < 0 \quad \text{for all} \quad \omega \in \mathbb{R} \quad \text{with} \quad |\omega| > \sqrt{3}.$$

**Example 3.** This example is taken from [29]. We consider the standard transfer function

$$h(s) = \mathbf{c}^{\mathrm{T}}\,(s\,\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{44}$$

Again, as in Example 2, $h$ has no poles and zeros in $\mathbb{C}_+$, yet $h$ is not positive real. Indeed, Algorithm 1 applied to (44) generates a matrix pencil $\mathbf{M} + \lambda\,\mathbf{N}$ that has one positive real eigenvalue $5.6295 \times 10^{15}$ with multiplicity 1.

**Example 4.** This is an example of a lossless positive real function taken from [3, p. 222]. We consider the standard transfer function

$$h(s) = \mathbf{c}^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}. \tag{45}$$

Algorithm 1 applied to (45) produces the transformed matrix

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

which has the eigenvalues $\pm i$ and $\pm i\sqrt{3}$, and the matrix pencil

$$\mathbf{M} + \lambda\,\mathbf{N} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 0 & -4 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since this pencil is singular, Algorithm 1 found that $h$ is lossless positive real.

Of course, for this simple example, lossless positive realness can also be verified by direct computation. Indeed, by [3, p. 222], the function (45) can be represented as follows:

$$h(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}. \tag{46}$$

By (46), $h$ has the poles $\pm i$ and $\pm i\sqrt{3}$, and $\operatorname{Re}\left(f(\mathrm{i}\,\omega)\right) = 0$ for all $\omega \in \mathbb{R}$. Thus $h$ is lossless positive real.

**Example 5.** The next example is taken from [3, p. 291]. We consider the transfer function

$$h(s) = \mathbf{c}^{\mathrm{T}} \left(s\,\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{b} + d, \quad \mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad d = \frac{1}{2}, \tag{47}$$

which has a nonzero directly-coupled term, $d \neq 0$. Therefore, as discussed in Section 3.3, we first rewrite (47) as a general transfer function:

$$g(s) = h(s) = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \left( s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \tag{48}$$

Note that, for $s = 0$, the inverse in (48) exists, and therefore we can set $s_0 = 0$ in Algorithm 2. With this shift, Algorithm 2 applied to (48) produces a pencil $\mathbf{S} - s\,\mathbf{T}$ with finite eigenvalues $-1$ and $\pm i$, and further the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ with finite eigenvalues $-3$ and $1 \pm 4.609e-08i$. Therefore, the function $h$ is positive real.

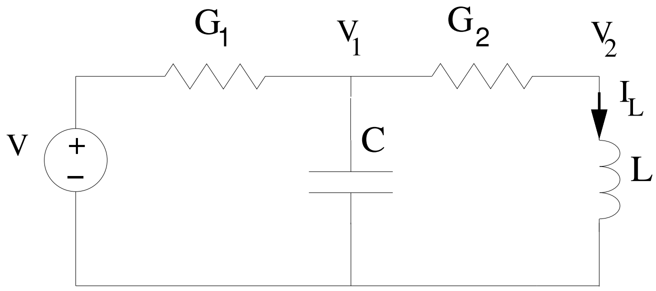


Figure 1: Small RLC circuit

For this example, since $d > 0$, as an alternative to Algorithm 2, we can also test positive realness via solving an algebraic Riccati equation; see, e.g., [9, Chapter 2]. More precisely, assuming that it has been verified that $\mathbf{A}$ has no eigenvalues with positive real part, then the function $h$ in (47) is positive real if, and only if, there exists a matrix $\mathbf{X} = \mathbf{X}^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ satisfying the algebraic Riccati equation

$$\mathbf{X}\,\mathbf{F} + \mathbf{F}^{\mathrm{T}}\,\mathbf{X} + \mathbf{X}\,\frac{1}{2d}\,\mathbf{b}\,\mathbf{b}^{\mathrm{T}}\,\mathbf{X} + \frac{1}{2d}\,\mathbf{c}\,\mathbf{c}^{\mathrm{T}} = \mathbf{0}. \tag{49}$$

Here, $\mathbf{F} := \mathbf{A} - \mathbf{b}\,\mathbf{c}^{\mathrm{T}}/(2d)$, and $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, and $d$ are given in (47). There are numerous algorithms for the solution of algebraic Riccati equations; see, e.g., [25] and the references given therein. For example, (49) can be converted to an eigenvalue problem for the Hamiltonian matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{F} & \frac{1}{2d}\,\mathbf{b}\,\mathbf{b}^{\mathrm{T}} \\ -\frac{1}{2d}\,\mathbf{c}\,\mathbf{c}^{\mathrm{T}} & \mathbf{F}^{\mathrm{T}} \end{bmatrix}. \tag{50}$$

The size of the matrix (50) is twice the size of the matrix pencil in our eigenvalue-based test method. As a result, even if the special structure of the matrix (50) is exploited, testing of positive realness via (49) or (50) is more expensive than Algorithm 2.

**Example 6.** Consider the small RLC circuit shown in Figure 1. The circuit consists of two resistors, one inductor, and one capacitor, and it is driven by a voltage source. This circuit can be described by the generalized transfer function

$$g(s) = \frac{V_1}{V} = \mathbf{e}_1^{\mathrm{T}}\,(s\,\mathbf{E} - \mathbf{A})^{-1}\,(G_1\mathbf{e}_1)\,, \tag{51}$$

where

$$\mathbf{A} = -\begin{bmatrix} G_1 + G_2 & -G_2 & 0 \\ -G_2 & G_2 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Here, $G_1$ and $G_2$ are the conductances of the two resistors, $L$ is the inductivity of the inductor, $C$ is the capacity of the capacitor, $V$ is the voltage generated by the voltage source, and $V_1$ is the voltage at one the circuit nodes; see Figure 1. Since any RLC circuit is passive, the function (51) is positive real.

For our numerical tests, we chose the values $G_1 = 1$, $G_2 = 0.1$, $C = 1$, $L = 1000$, and then ran Algorithm 2 to check if (51) is indeed positive real. Since $\mathbf{A}$ is nonsingular, we can use the shift $s_0 = 0$. Algorithm 2 finds that $g$ has two finite poles at $-0.99899$ and $-0.01101$. The finite eigenvalues of the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ are $-1.1 \times 10^{-4}$ and $-2.5 \times 10^{26}$. Moreover, $g(s_0) = \mathbf{l}^{\mathrm{T}}\mathbf{r} > 0$, and $\det(\mathbf{M} + \lambda_0\,\mathbf{N}) = 9091.9 > 0$ for $\lambda_0 = 1$.

**Example 7.** This example is taken from [6]. It arises in the analysis of a package model used for an RF integrated circuit. The package model is replaced by a reduced-order model generated by means of the PVL method [14]. Algorithm 2 is used to check if the model produced by $n = 80$ PVL iterations is passive, i.e., if its transfer function is positive real. The reduced-order transfer function is given by

$$g_{80}(s) = \mathbf{l}^{\mathrm{T}}\mathbf{r}\;\mathbf{e}_1^{\mathrm{T}}\left(\mathbf{S}_{80} - s\,\mathbf{T}_{80}\right)^{-1}\mathbf{e}_1,$$

where $\mathbf{T}_{80}$ is the $80 \times 80$ Lanczos matrix, $\mathbf{S}_{80} = \mathbf{I} + s_0\,\mathbf{T}_{80}$, and $s_0 > 0$ is a suitably chosen expansion point. The approximation $g_{80}$ is sufficiently accurate in the frequency range of interest, but has 2 poles and 4 zeros in $\mathbb{C}_+$. Therefore, the reduced-order model given by $g_{80}$ is not passive.

Next, we apply the post-processing PVL$\pi$ technique [6] to $g_{80}$ to force the unstable poles and zeros into the left half of the complex plane. The modified transfer function produced by PVL$\pi$ is of the form

$$\widehat{g}_{80}(s) = \mathbf{l}^{\mathrm{T}}\mathbf{r}\;\mathbf{e}_1^{\mathrm{T}}\left(\widehat{\mathbf{S}}_{80} - s\,\widehat{\mathbf{T}}_{80}\right)^{-1}\mathbf{e}_1.$$

Here, $\widehat{\mathbf{S}}_{80} = \mathbf{I} + s_0\,\widehat{\mathbf{T}}_{80}$, $\widehat{\mathbf{T}}_{80} = \mathbf{T}_{80} + \mathbf{x}\,\mathbf{e}_n^{\mathrm{T}}$, and $\mathbf{x}$ is a correction vector computed by PVL$\pi$. All poles and zeros of $\widehat{g}_{80}$ now lie in the left half of the complex plane.

To verify the passivity of $\widehat{g}_{80}(s)$, it only remains to execute Steps 7–10 of Algorithm 2. One finds that the pencil $\mathbf{M} + \lambda\,\mathbf{N}$ has no real positive eigenvalues and that $\mathbf{l}^{\mathrm{T}}\mathbf{r}\;\det(\mathbf{M} + \lambda_0\,\mathbf{N}) > 0$ for $\lambda_0 = 1$. Therefore, Algorithm 2 confirmed that the reduced-order model given by $\widehat{g}_{80}$ is indeed passive.

## 6 Concluding Remarks

We have presented an eigenvalue-based characterization of positive realness of single-input single-output time-invariant linear systems. This characterization leads to efficient numerical procedures that involve only standard matrix computations and thus can be easily implemented in Matlab or by using off-the-shelf linear algebra libraries, such as LAPACK [4]. The total computational cost is of the order $n^3$ for a single-input single-output system of state-space dimension $n$. Numerical examples from simple model problems to applications in reduced-order modeling of VLSI circuits demonstrate the simplicity and effectiveness of these eigenvalue-based test procedures.

The eigenvalue-based tests developed in this paper are different from the usual Routh-Hurwitz like procedures and from the tests derived from the PR Lemma. On the other hand, the PR Lemma is more general in that it can be employed for general multi-input multi-output systems. It is subject of a future study to derive extensions of our eigenvalue-based tests to multiple-input multiple-output time-invariant linear systems.

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