Vertex-rounding a three-dimensional polyhedral subdivision

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Abstract

Let P be a polyhedral subdivision in $\mathbb{R}^3$ with a total of n faces. We show that there is an embedding σ of the vertices, edges, and facets of P into a subdivision Q, where every vertex coordinate of Q is an integral multiple of $2^{-\lceil \log_2 n+2 \rceil}$. For each face f of P, the Hausdorff distance in the L_∞ metric between f and $\sigma(f)$ is at most 3/2. The embedding σ preserves or collapses vertical order on faces of P. The subdivision Q has $O(n^4)$ vertices in the worst case, and can be computed in the same time.

1 Introduction

Geometric algorithms are usually described in the "real-number RAM" model of computation, where arithmetic operations on real numbers have unit cost. A programmer implementing a geometric algorithm must find some substitution for real arithmetic. The substitution of exact arithmetic on a subset of the reals, say the integers or the rationals, avoids the difficulties that can arise from naive substitution of floating-point arithmetic [4, 12, 14, 15]. The substitution is not trivial, since the required arithmetic bit-length usually exceeds the native arithmetic bit-length of most computer hardware, and some form of software arithmetic is required.

Recent research has made the use of software exact arithmetic for geometric algorithms much more attractive. A predicate on geometric data is determined by the sign of an arithmetic expression in the coordinates of the data. A promising strategy for sign-evaluation is adaptive-precision arithmetic [6, 13, 20], where the expression is evaluated to higher and higher precision until its sign is known, i.e. until the magnitude of the expression exceeds an error bound. Low precision, even floating-point, suffices most of the time, since most instances of geometric predicates are easy. In addition, for some basic predicates like the sign of a determinant, there are alternative evaluation strategies that require arithmetic with relatively low precision [1, 2, 3].

Exact arithmetic would be more useful if high-level geometric rounding algorithms were available. Virtually any geometric construction that produces new geometric data increases the bit-length of geometric coordinates. For example, suppose points are represented with homogeneous integer coordinates. The plane through three such points has coefficients whose bit-lengths are about three times the point coordinate bit-lengths; the point of intersection of three such planes has coordinate bit-length about nine times that of the original points. Thus a solid modeler, which implements boolean operations and rigid motions on polyhedra, might produce a polyhedron with high coordinate bit-length even if the original polyhedra had short coordinate bit-length. Typically an application requires only a low-precision approximation, not the exact answer. Hence there is a need for high-level rounding, which replaces a geometric structure with high bit-length coordinates with an approximating structure with short bit-length coordinates. It does not suffice to round each coordinate independently, since such rounding is a geometric perturbation, and may introduce inconsistencies between geometric and combinatorial information. Furthermore, some change in combinatorial structure is inevitable; indeed, in certain cases it is NP-hard to determine if it is possible to round to low-precision without changing combinatorial structure [19].

Satisfactory high-level rounding algorithms are known for polygonal subdivisions in two dimensions. One such algorithm is snap-rounding [10]. Fix a polygonal subdivision, with arbitrary-precision coordinates. A *pixel* is a unit square in the plane centered at a point with integer coordinates; a pixel is *hot* if it contains a vertex of the subdivision. Snap-rounding replaces each vertex by the center of the pixel containing the vertex, and each edge by the polygonal chain through the centers of the hot pixels met by the edge, in the same order as met by the edge. The snap-rounded subdivision approximates the original subdivision in the sense that each vertex and edge of the original subdivision has an image in the snap-rounded arrangement whose Hausdorff distance is at most 1/2 in the L_∞ metric. Snap-rounding may change the combinatorial structure of the subdivision, for example, vertices and edges may collapse together, but some combinatorial ordering information is preserved [10].

This paper presents a generalization of snap-rounding to polyhedral subdivisions in three dimensions. Fix a polyhedral subdivision P with a total of n vertices, edges, and facets. We show that there is a polyhedral subdivision Q so that each vertex coordinate is an integer multiple of $1/2^{\lceil \log_2 n \rceil + 2}$. Each face f of P has an image $\sigma(f)$ in Q so that the Hausdorff distance between f and $\sigma(f)$ is at most 3/2. As with snap-rounding in two dimensions, f and $\sigma(f)$ may have different combinatorial structure: an edge may be replaced with a polygonal chain, and a facet with a triangulation. Two vertices may collapse together; the polygonal chains for two edges or the triangulations for two facets may collapse together or overlap partially, perhaps in several places. However, vertical order is preserved (or collapsed): if face f is vertically above face f' (i.e. there is a line parallel to the z-axis meeting both faces, and the intersection with f has higher z-coordinate), then $\sigma(f)$ is above (or overlaps) $\sigma(f')$. In the worst case Q has $O(n^4)$ vertices and can be computed in time $O(n^4)$.

As is the case with snap-rounding in two dimensions, the Hausdorff distance between a facet f and its image $\sigma(f)$ can be reduced by scaling coordinates. For example, for $k > 0$, the Hausdorff distance can be reduced to at most $\frac{3}{2} \cdot 2^{-k}$, by multiplying every coordinate of P by 2^k, rounding as above, and then dividing every coordinate of the result by 2^k. Of course, coordinates are now integral multiples of $1/2^{k+\lceil \log n \rceil + 2}$. Alternatively, the same procedure with $k = -(\lceil \log n \rceil + 2)$ guarantees an approximating subdivision with integer coordinates, although with Hausdorff distance bounded by $6n$.

Though the algorithm in this paper demonstrates the theoretical possibility of three-dimensional snap-rounding, it is not immediately practical. One concern is the discouragingly large bound on the number of new vertices. The algorithm as described always adds all vertices that might potentially be needed; a variant algorithm might add vertices only as necessary. It is plausible that for typical subdivision instances–not chosen by an adversary–the number of new vertices will be acceptable. Another concern is the complexity of the algorithm. Snap-rounding in two dimensions is essentially a local algorithm, with the rounding of each vertex and edge determined simply from the set of hot pixels. Unfortunately, as is seen below, the three-dimensional algorithm requires more complicated global information. Devising a simple, practical, and efficient three-dimensional rounding algorithm is a significant open problem.

Other work. Greene and Yao were the first to suggest a rounding scheme for polygonal subdivisions in two dimensions [8]. Hobby [11] and Greene [9] give algorithms to compute the snap-rounding of the arrangement formed by a set of intersecting edges. Guibas and Marimount [10] show how to maintain the snap-rounded arrangement of edges under insertion and deletion of edges; they also give elementary proofs of basic topological properties of snap rounding. Goodrich *et al* [7] give improved algorithms to snap-round a set of intersecting edges, in the case when there are many intersections within a pixel. Milenkovic [18] suggests a "shortest-path" geometric rounding scheme that sometimes introduces fewer bends than snap rounding.

Goodrich *et al* [7] propose a scheme for snap-rounding a set of edges in three dimensions after first adding as vertices the points of "closest encounter" between nearby edges. Milenkovic [16] sketches a scheme for rounding a polyhedral subdivision in three dimensions (in fact, any dimension). Unfortunately, both schemes have the property that rounded edges can cross (see below), which violates any notion of topological consistency.

Fortune [5] suggests a high-level rounding algorithm for polyhedra in three dimensions. His algorithm assumes that a polyhedron is presented by the equations of its face planes (and the combinatorial incidence structure of faces), not the coordinates of vertices as assumed by snap-rounding. His algorithm does not appear to extend from polyhedra to polyhedral subdivisions.

The challenges of 3d snap-rounding. The obvious way to snap-round a vertex in three dimensions is to replace it with the center of the voxel containing it. (A *voxel* is a unit cube centered at an integer point.) It is less clear how to snap-round edges and facets.

Snap-rounding a set of edges in three dimensions requires the addition of new vertices, unlike the situation in two dimensions. Consider two transverse nearby edges. Rounding the endpoints to voxel centers perturbs the edges, and hence the edges may change orientation or cross. We can attempt to prevent this by adding a vertex in the interior of each edge near the other edge; then either the two new vertices are in the same voxel and snap-round together, or they are in different voxels and the snap-rounded edges will not cross. Clearly, it might be necessary to add quadratically many vertices, if the edges form a "cross-hatch" pattern.

Snap-rounding with facets as well is more problematic. If a vertex v and a facet f are nearby, we can add a new vertex v' to f to ensure that v and f are properly separated or collapsed. However, this requires that f be triangulated, which introduces new edges. Potentially these edges are close to old edges, which could require new vertices, and it is not immediate that the process is finite. We can attempt to ensure termination by projecting nearby edges onto a facet, and then triangulating the facet compatibly with the projection. The actual rounding algorithm is a formalization of this idea.

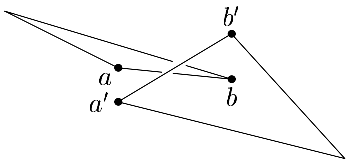

Figure 1: Vertices a and a' project to the same pixel in the xy-plane, as do b and b'. Hence in three dimensions, the snap-rounding of ab crosses the snap-rounding of $a'b'$.

Overview of the rounding algorithm. The rounding algorithm is based on the following general outline. Orthogonally project all edges of the subdivision P onto the xy-plane, form the arrangement, snap-round, and compute a triangulation T. Each facet f of P has an image T_f that forms a subtriangulation T_f of T. The rounding of facet f is obtained by lifting T_f to a polygonal surface $\sigma(f)$ that approximates f. By considering each cylinder over a vertex, edge, or triangle of T separately, we can ensure that the lifting preserves (or collapses) the vertical order on faces of P.

The first step of the actual algorithm is to determine the rounding $\sigma(e)$ of each edge e of P; recall that $\sigma(e)$ in general can be a polygonal chain. This step is nontrivial, since we must prevent crossings among the resulting edges (see Figure 1). To prevent crossings, we subdivide the edges of P by all $xy-$, $xz-$, and yz-intersection points. (If the orthogonal projections of e and e' into the xz-plane cross at a point p, and l is the line parallel to the y-axis through p, then $e \cap l$ and $e' \cap l$ are *xz-intersection points.*) While this subdivision prevents most crossings, it is not quite sufficient to prevent all crossings. In figure 2, the xy-, xz-, and yz-projections of d^* and e^* are all disjoint, but their snap-roundings cross. Fortunately, the configuration of figure 2 is almost the only way this can happen, and we can show that there is a slight modification of snap-rounding that does avoid crossings. For example, in figure 2, the modified snap-rounding of d^* is a two-edge polygonal chain, connecting a snap-rounded endpoint of d^* to the snap-rounded endpoint of e^* on the same vertical line, and then to the other snap-rounded endpoint of d^*. We show that the distance between an edge and its modified snap-rounding increases slightly, to at most 3/2. (The configuration in figure 2 can be modified to show that the "close encounter" subdivision of Goodrich *et al* [7] does not prevent

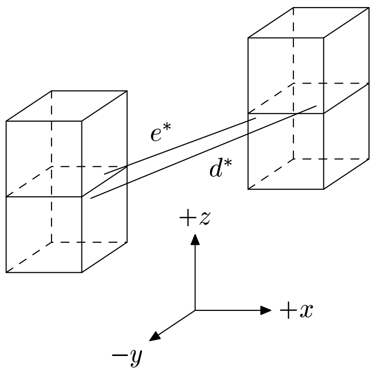

Figure 2: The endpoints of e^* and d^* lie on column boundaries (or extend slightly inside). The roundings $\rho(d^*)$ and $\rho(e^*)$ cross, although the xy-, xz-, and yz-projections of d^* and e^* do not.

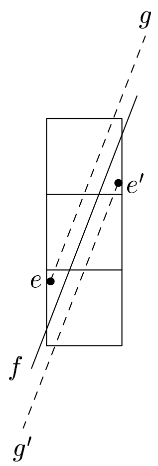

Figure 3: Side view. Edge e is above facet f and e' is below. Hence the rounding of facet f must contain the vertical interval from the rounding of e to the rounding of e'.

edge crossings.) Section 4 below describes the subdivision and modification of snap-rounding.

The second step of the actual algorithm is to lift T_f to its image $\sigma(f)$. The lifting must respect constraints on vertical order among facets and edges; for example, if facet f is vertically above edge e, then $\sigma(f)$ must be above or contain $\sigma(e)$. These constraints can be somewhat complex. In the schematic view in figure 3; facet f is below edge e of facet g and above edge e' of facet g'. If $\sigma(e)$ and $\sigma(e')$ have the same xy-projection with $\sigma(e')$ above $\sigma(e)$, then $\sigma(f)$ must contain the entire vertical interval between $\sigma(e)$ and $\sigma(e')$ (as do $\sigma(g)$ and $\sigma(g')$, in this case).

The lifting $\sigma(f)$ is determined by merging the lifted images of each vertex, edge, and triangle of T_f. For a vertex v of T_f, its lifting $l_f(v)$ may just be a vertex; however it could be a vertical chain of edges if there are vertices v' and v'' of P so that v' is above f, v'' is below f, $\sigma(v')$ is below $\sigma(v'')$, and both $\sigma(v')$ and $\sigma(v'')$ project and snap-round to v. Similarly, the lifting $l_f(e)$ of an edge e of T_f may just be an edge or it may contain the vertical interval between two edges whose xy-projection snap-rounds to e.

The lifting $l_f(\Delta)$ of a triangle Δ of T_f is more complicated. It is defined in terms of the lifting $l_{f\Delta}(e)$ for each edge e of Δ. The lifting $l_{f\Delta}(e)$ is just an edge with xy-projection e; it will form part of the boundary of $l_{f\Delta}(e)$. It must satisfy three properties: it must be close to f, it must not cross any other lifted edge, and it must respect vertical order with other lifted edges, (i.e. if f is vertically below f' , then $l_{f\Delta}(e)$ must be below $l_{f'\Delta}(e)$). The last property is crucial to establishing that the lifted triangles $l_f(\Delta)$ respect vertical order. Satisfying all three properties requires some care (see section 5).

There are two naively-plausible properties of the definition of $l_{f\Delta}$ that do not hold. First, if Δ and Δ' are both incident to e and in T_f, then there is no guarantee that $l_{f\Delta}(e) = l_{f\Delta'}(e)$ (though they don't cross). However, this causes no difficulty (since $l_f(e)$ contains both $l_{f\Delta}(e)$ and $l_{f\Delta'}(e)$).

The second untrue property causes more difficulty. Suppose Δ has vertices a, b, c. There is no guarantee that, say, $l_{f\Delta}(ab)$ is incident to $l_{f\Delta}(ac)$ (though both meet the vertical line through a). Hence $l_f(\Delta)$ must be a triangulation of the polygon formed by $l_{f\Delta}(ab)$, $l_{f\Delta}(bc)$, and $l_{f\Delta}(ac)$, and

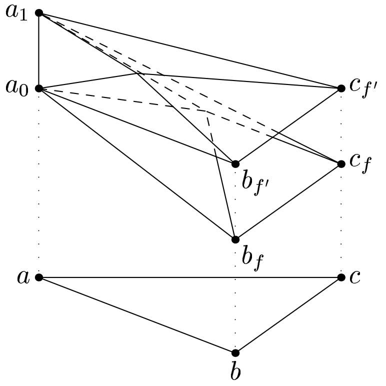

Figure 4: The liftings of triangle Δabc for facets f and f' have boundary $a_1 a_0 b_f c_f$ and $a_1 a_0 b_{f'} c_{f'}$, respectively.

perhaps edges along the vertical lines through a, b, and c. See figure 4. It is easy to triangulate the polygon using a central vertex whose xy-projection is within triangle Δ. However, a vertical boundary edge may be shared among several different liftings. To ensure that there are no crossings among edges, each central vertex must have distinct coordinates. Since there may be $\Omega(n)$ central vertices, coordinates that are integer multiples of roughly $1/n$ are necessary. This leads to the additional $\lceil \log n \rceil + 2$ bits needed for vertex coordinates. More details of the lifting appear in section 6.

Naively the rounded subdivision Q has at most $O(n^3)$ faces: the triangulation T has $O(n^2)$ triangles, so for each facet f the rounding $\sigma(f)$ consists of $O(n^2)$ lifted triangles $\{l_f(\Delta)\}$. However, in the worst case each lifted triangle $l_f(\Delta)$ may consist of $O(n)$ faces, since there could be linearly many vertices on the vertical edges of its boundary. Hence Q has $O(n^4)$ faces.

2 The main theorem

For points $a, b \in \mathbb{R}^3$ and sets $A, B \subset \mathbb{R}^3$, $d(a,b)$ is the L_∞ distance between a and b (the L_∞ distance is used exclusively in this paper); $d(a,B)$ is $\inf_{b \in B} d(a,b)$; and $d(A,B)$ is $\sup_{a \in A} d(a,B)$. Note that d is symmetric for points, but not in general for sets. *Hausdorff distance* $d_H(A,B)$ is $\max(d(A,B), d(B,A))$.

The direction parallel to the z-axis is the *vertical* direction. Two sets $A, B \subset \mathbb{R}^3$ are *vertically ordered* $A \prec B$ (read "A is below B") if there is a vertical line meeting both A and B, and for every vertical line l meeting A and B, $A \cap l$ is below $B \cap l$, i.e. the z-coordinate of every point of $A \cap l$ is less than the z-coordinate of every point in $B \cap l$. Sets A and B satisfy $A \preceq B$ if there is a vertical line meeting both, and for every vertical line meeting both, $A \cap l$ is below or intersects $B \cap l$. As is well-known, $\prec$ is not transitive in general; it is transitive among a family of sets that have the same xy-projection. If furthermore every family in the set is a *surface*, i.e. every vertical line misses the set or meets it at one point, then $\preceq$ is transitive.

A *subdivision* P in $\mathbb{R}^3$ is a set of compact convex polyhedral *cells* so that every face of every cell is in the subdivision and so that the intersection of two cells is a face of both. Cells of dimension 0, 1, and 2 are *vertices*, *edges*, and *facets*, respectively. $|P|$ is the union of the cells of P. An *embedding* of a subdivision P into a subdivision Q is a mapping σ that maps each cell of P into a subdivision contained in Q so that if f is a face of f', then $\sigma(f) \subseteq \sigma(f')$.

To simplify notation somewhat, we extend d and $\prec$ to subdivisions. Thus for subdivisions P and Q, $P \prec Q$ means $|P| \prec |Q|$ and $d(P,Q)$ means $d(|P|,|Q|)$.

Throughout this paper we assume that subdivisions in $\mathbb{R}^3$ do not include cells of dimension 3. Furthermore, we assume that every subdivision is in general position, specifically, that no edge or facet is parallel to a coordinate axis and that no vertex has a coordinate that is an integer multiple of $1/2$. The general position assumption simplifies presentation; it is not hard to remove (either explicitly or for example by an infinitesimal symbolic rigid motion).

Theorem 2.1 *Let P be a subdivision in $\mathbb{R}^3$ with a total of n cells; set $\kappa = 3/2$. There is a subdivision Q and an embedding σ of P into Q so that:*

1. *For each cell f of P, $d_H(f, \sigma(f)) < \kappa$.*
2. *Each vertex coordinate of Q is an integral multiple of $1/2^{\lceil 2+\log_2 n \rceil}$.*
3. *If cells f, f' of P satisfy $f \preceq f'$, then $\sigma(f) \preceq \sigma(f')$.*
4. *Q can be computed in time $O(n^4)$ and has $O(n^4)$ cells.*

This theorem follows from the discussion below, in particular results 6.2, 6.5, 6.6, and 7.4 below. At a high level, the algorithm required for step (4) has three steps.

1. Subdivide the vertices and edges of P, forming a set of vertices and edges P^* (Section 4).
2. Orthogonally project P^* onto the xy-plane, snap-round, and triangulate the convex hull of the resulting subdivision. Let T be the resulting triangulation.
3. For each cell f in P, lift T_f (the image of f in T) to a subdivision $Q_f \subset \mathbb{R}^3$ (Section 6).

3 Definitions

There are many symbols defined in this paper. For reference, most are summarized in appendix B.

A *pixel* is an open unit square in the xy-plane centered at an integer point; pixel(q) is the pixel containing point q. A *voxel* is an open unit cube in $\mathbb{R}^3$ centered at an integer point; voxel(q) is the voxel containing point q. A *column* (of voxels) is all voxels whose centers have the same x- and y-coordinates; column(q) is the column containing q.

Let A be a subdivision in the xy-plane. A pixel is *hot* (with respect to A) if it contains a vertex. The *snap-rounding* (with respect to A) of an edge e of A is the polygonal chain connecting the centers of the hot pixels met by e in the same order as met by e; similarly the *snap-rounding* of a vertex of A is the center of the hot pixel containing it. A basic fact[10] is that two polygonal chains that result from snap-rounding intersect only at vertices and edges of both chains. The *snap-rounding* of A is obtained by replacing each edge and vertex of A with its snap-rounding with respect to A; it is a polygonal subdivision whose vertices are hot pixel centers, i.e. integer points, and whose edges connect integer points.

Let π_{xy} be orthogonal projection onto the xy-plane, and similarly for π_{xz} and π_{yz}. A set $A \subset \mathbb{R}^3$ is *over* a set P in the xy-plane if $\pi_{xy}(A) = P$. If A is a surface with $p \in \pi_{xy}(A)$, then A_p is the point of A over p (i.e. $\pi_{xy}(A_p) = p$). If A and B are surfaces over the same set, then $\max(A, B)$ is the pointwise maximum (viewed as functions of the xy-plane), and $\min(A, B)$ is the pointwise minimum. If A, B, C are surfaces over the same set with $A \succeq B$, then $\text{snap}(C, [A, B])$ is $\min(A, \max(B, C))$. Clearly, $A \succeq \text{snap}(C, [A, B]) \succeq B$.

Suppose a set P in the xy-plane is fixed. We define symbolic sets $\top$ (top) and $\perp$ (bottom) satisfying $\perp \prec A \prec \top$ for any other set A over P. We have for example $\min(A, \top) = A = \max(A, \perp)$; we define min and max of an empty collection to be $\top$ and $\perp$, respectively.

Two edges *cross* if they intersect at a point interior to at least one of the edges.

Proposition 3.1 *Suppose $T \subset \mathbb{R}^3$ is convex, $\{s_1, \ldots, s_k\} \subset \mathbb{R}^3$ is a finite set of points with convex hull S, and $\kappa \geq 0$. If $d(s_i, T) \leq \kappa$ for $i = 1, \ldots, k$, then $d(S, T) \leq \kappa$.*

Proof: Any point in S can be expressed as $\sum \alpha_i s_i$ with $0 \leq \alpha_i \leq 1$ and $\sum \alpha_i = 1$. For each s_i, there is a point $t_i \in T$ so that $d(s_i, t_i) \leq \kappa$. Clearly $\sum \alpha_i t_i \in T$ and $d(\sum \alpha_i s_i, \sum \alpha_i t_i)$ is the maximum absolute value of any coordinate of $\sum \alpha_i(s_i - t_i)$, which is bounded by κ since $\sum \alpha_i = 1$, $\alpha_i \geq 0$, and the absolute value of each coordinate of $s_i - t_i$ is bounded by κ. $\square$

4 Snap-rounding edges

Define $\rho(q)$ to be the center of the voxel containing q, and extend to ρ to edges: $\rho(qq')$ is the edge $\rho(q)\rho(q')$. The mapping ρ is the obvious extension of snap-rounding to three dimensions (ignoring snapping to hot voxels, which is unimportant here). Unfortunately, ρ may cause two edges to cross. We now define a refinement P^* of the vertices and edges of P and a modification τ of ρ so that no two edges in $\tau(P^*)$ cross.

4.1 The subdivision P^*

Let e and e' be two edges of P whose xy-projections cross at a point p. An *xy-intersection point* (of P) is either point on e or e' that meets the line through p parallel to the z-axis. The definition of an $xz-$ or yz-intersection point is similar.

The subdivision P^* results from subdividing the edges of P. At any point in the process, $\hat{e}$ denotes the subdivision of edge e of P; any voxel containing a vertex is a *hot* voxel; and any column of voxels containing a hot voxel is a *hot* column. There are two steps in the subdivision:

1. Subdivide the edges of P at all xy-, xz-, and yz-intersection points of P.

2. For each edge e of P, split $\hat{e}$ by each hot column C it meets: $\hat{e}$ must meet C in a consecutive set of voxels; $\hat{e}$ is *split* by C by further subdividing $\hat{e}$ at any point in the first voxel (if $\hat{e}$ does not yet have a vertex in the first voxel) and similarly by subdividing $\hat{e}$ in the last voxel.

Splitting by hot columns has an easy consequence: for any edge e of P, the snap-rounding of $\pi_{xy}(\hat{e})$ with respect to $\pi_{xy}(P^*)$ is identical to the snap rounding of $\pi_{xy}(\hat{e})$ with respect to $\hat{e}$. Henceforth we use a superscript '*' for edges and vertices of P^*. For e^* an edge of P^*, we write $s(e^*)$ for the snap-rounding of $\pi_{xy}(e^*)$. It is immediate that if d^*, e^* are edges of P^*, then $\rho(d^*)$ crosses $\rho(e^*)$ only if $s(d^*) = s(e^*)$.

Lemma 4.1 *P^* has $O(n^3)$ vertices; there are $O(n^2)$ hot columns and $O(n^3)$ hot voxels.*

Proof: Clearly there are at most $O(n^2)$ xy-, xz-, and yz-intersection points, and only $O(n)$ vertices of P. Splitting edges by hot columns adds no new hot columns, hence there are $O(n^2)$ hot columns. For each edge e of P and for each hot column, there are at most two vertices added when $\hat{e}$ is split by the column. Hence there are $O(n^3)$ vertices altogether. □

As mentioned earlier, T is a triangulation of the convex hull of $s(P^*)$. Consider the edges E^* in P^* bounding a facet f of P. The projection $\pi_{xy}(E^*)$ forms a simple cycle, but the snap-rounding $s(E^*)$ need not. However, it is not hard to see that $s(E^*)$ consists of some number of simple cycles connected by polygonal chains. Let T_f be the subtriangulation of T consisting of the vertices and edges of $s(E^*)$ plus any vertices, edges and triangles of T interior to the simple cycles in $s(E^*)$.

For v a vertex of T, e an edge of T, and Δ a triangle of T, define

$$\begin{aligned} P_e^* &= \{e^* \in P^* : s(e^*) = e\} \\ P_v^* &= \{v^* \in P^* : s(v^*) = v\} \\ F_e &= \{f \in P^* : e \in T_f\} \\ F_\Delta &= \{f \in P^* : \Delta \in T_f\} \end{aligned}$$

where v^* and e^* are vertices and edges of P^*, respectively, and f is a facet of P.

4.2 The mapping τ

Lemma 4.2 *Let e be an edge of T. If $d^*, e^* \in P_e^*$ and $\rho(d^*), \rho(e^*)$ cross, then either there is an endpoint w of $\rho(d^*)$ with $d(w, e^*) < \kappa$ or an endpoint w' of $\rho(e^*)$ with $d(w', d^*) < \kappa$.*

The proof of this lemma is rather intricate, so it is deferred to the appendix.

Lemma 4.3 *Let e be an edge of T. There is a mapping τ defined on P_e^* so that*

1. *For all edges $e^* \in P_e^*$, $\tau(e^*)$ is an edge over e with endpoints among the endpoints of $\rho(P_e^*)$.*
2. *For all edges e^*, $d(\tau(e^*), e^*) < \kappa$.*
3. *$\tau(P_e^*)$ is noncrossing.*

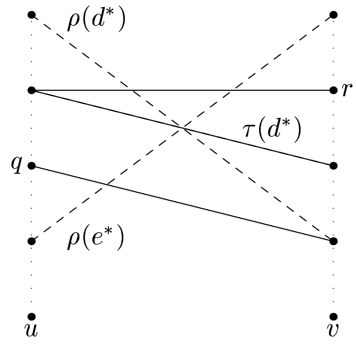

Figure 5: Definition of τ on new edge e^*.

4. τ can be computed in time quadratic in the size of P_e^*.

Proof: We define τ inductively, adding edges of P_e^* one by one in arbitrary order. The addition of an edge may change the definition of τ on other edges as well; however, properties (1) through (3) of the lemma statement are maintained. For the following, recall that A_p is the point of A that has xy-projection p.

So suppose τ has been defined on a subset S of P_e^* and e^* is the next edge. If no edge of $\tau(S)$ crosses $\rho(e^*)$, then simply define $\tau(e^*) = \rho(e^*)$. Otherwise, since $\tau(S)$ is noncrossing, we can assume up to a symmetric argument that every edge $\tau(d^*)$ crossing $\rho(e^*)$ has $\tau(d^*)_u \succ \rho(e^*)_u$ and $\tau(d^*)_v \prec \rho(e^*)_v$.

Let q be the highest (in $\prec$) endpoint over u of an edge in $\tau(S) \cup \{\rho(e^*)\}$ so that $d(q, e^*) < \kappa$; similarly let r be the lowest endpoint over v of an edge in $\tau(S) \cup \{\rho(e^*)\}$ so that $d(q, e^*) < \kappa$. Clearly $\rho(e^*)_u \preceq q$ and $r \preceq \rho(e^*)_v$. If it is is possible to choose q' over u in the interval $\rho(e^*)_u \ldots q$ and r' over v in the interval $r \ldots \rho(e^*)_v$ so that $q'r'$ does not cross an edge of $\tau(S)$, define $\tau(e^*)$ to $q'r'$. Note that it is always possible to choose q' and r' among the endpoints of $\tau(S) \cup \{\rho(e^*)\}$ and that the distance from q' and r' to e^* is less than κ.

Otherwise some subset S' of the edges in S crosses qr. Clearly for any $d^* \in S'$, $\tau(d^*)_u \succ q$ and $\tau(d^*)_v \prec r$. See figure 5.

We claim that for any edge $\tau(d^*) \in S'$, either $d(q, d^*) < \kappa$ or $d(r, d^*) < \kappa$. If $\rho(d^*)_u \preceq q$, then since $q \preceq \tau(d^*)_u$, certainly $d(q, d^*) < d(\tau(d^*)_u, d^*) < \kappa$. Similarly if $\rho(d^*)_v \succeq r$, then $d(r, d^*) < \kappa$. Otherwise $\rho(d^*)_u \succ q \succeq \rho(e^*)_u$ and $\rho(d^*)_v \prec r \preceq \rho(e^*)_v$, so $\rho(d^*)$ crosses $\rho(e^*)$. See figure 5. The hypothesis of Lemma 4.2 holds with d^* and e^*. By the definitions of q and r, $d(\rho(d^*)_u, e^*) \geq \kappa$ and $d(\rho(d^*)_v, e^*) \geq \kappa$. Hence lemma 4.2 implies either $d(\rho(e^*)_v, d^*) < \kappa$ or $d(\rho(e^*)_u, d^*) < \kappa$, so either $d(q, d^*) < \kappa$ or $d(r, d^*) < \kappa$.

Let Q be the set of edges $d^* \in S'$ so that $d(q, d^*) < \kappa$, and $R = S' \setminus Q$. Define $\tau(e^*) = qr$; for $d^* \in Q$, redefine $\tau(d^*)_u = q$; and for $d^* \in R$, redefine $\tau(d^*)_v = r$. It is easy to check that τ satisfies conditions (1) through (3). The running time is immediate. $\square$

Henceforth we let τ be defined on all edges of P^*, by choosing a definition on P_e^* separately for each edge e of T, using lemma 4.3. Since there can be $O(n^2)$ edges e in T, and $O(n)$ edges in P_e^*, computation of τ takes time $O(n^4)$.

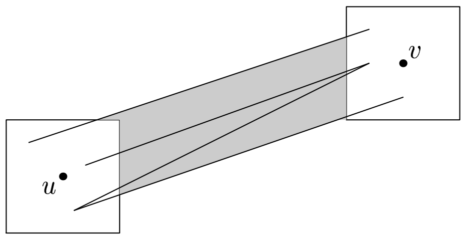

Figure 6: R_e is the shaded region plus the portion of the edges inside pixel(u) and pixel(v).

We remark that there is no guarantee that $\tau(e^*)$ and $\rho(e^*)$ have the same endpoints or indeed that $d(e^*, \tau(e^*)) \leq \kappa$. In section 6, we guarantee both properties by in effect augmenting $\tau(e^*)$ to a polygonal chain using vertical edges connecting its endpoints to the endpoints of $\rho(e^*)$.

5 Lifting triangle edges

The rounding $\sigma(f)$ of a facet f is defined from a lifting l_f of the vertices, edges, and triangles of T_f. The definitions of σ and l_f will appear in section 6 below. This section defines an auxiliary lifting function $l_{f\Delta}(e)$ required to define $l_f(\Delta)$. For e an edge of $\Delta \in T_f$, $l_{f\Delta}(e)$ is an edge over e; it will form part of the boundary of $l_f(\Delta)$.

The lifted edges $\{l_{f\Delta}(e) : f \in F_\Delta\}$ must satisfy three properties: each edge $l_{f\Delta}(e)$ must be close to f; the edges must respect vertical order (i.e. $f \preceq f'$ must imply $l_{f\Delta}(e) \preceq l_{f'\Delta}(e)$), and no pair of lifted edges may cross. These properties are the main result of this section (lemma 5.3). The definition of $l_{f\Delta}(e)$ and the proof of the properties are rather technical; on first reading it may be appropriate to skip to section 6.

Section 5.1 below gives important technical tools for the rest of the section: "covering order" on the facets in F_Δ and the "snapping lemma." Covering order is used to order the choices of $\{l_{f\Delta}(e) : f \in F_\Delta\}$. Suppose facet f follows facet f' in covering order and $l_{f'\Delta}(e)$ has been chosen to be close to f'. If, say, $f \prec f'$, then the snapping lemma guarantees that it is possible to choose $l_{f\Delta}(e)$ so that both $l_{f\Delta}(e) \preceq l_{f'\Delta}(e)$ and $l_{f\Delta}(e)$ is close to f. In order to make the snapping lemma appropriately transitive, it is necessary to have a careful definition of what it means for an edge to be close to a facet ("the edge approximates the facet").

5.1 The order $\lhd$ and the snapping lemma

Let edge e of T have endpoints u and v. Define R_e to be the convex hull of $\pi_{xy}(P_e^*)$, less the interior of pixel(u) and pixel(v), unioned with $\pi_{xy}(P_e^*)$. See figure 6. Notice that there are no intersections among the boundaries of $\{\pi_{xy}(f) : f \in F_\Delta\}$ within R_e except possibly at the endpoints of edges of $\pi_{xy}(P_e^*)$. Facet $f \in F_e$ *covers* e if no edge in P_e^* bounds f; it is easy to check that $R_e \subseteq \pi_{xy}(f)$. A facet f *covers facet* f' *at* e if $\pi_{xy}(f') \cap R_e \subseteq \pi_{xy}(f) \cap R_e$. For any two facets $f, f' \in F_e$, either f covers f' at e, or f' covers f at e.

Suppose that e is an edge of triangle Δ of T. The *covering order* $\lhd$ on the facets in F_Δ is any

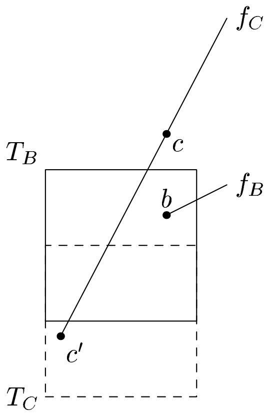

Figure 7: Proof of lemma 5.1, side view. T_B is solid square, T_C dashed.

total order so that $f \lhd f'$ implies f' covers f at e. (The order depends on both e and Δ, but to keep the notation simple we do not make this dependence explicit.) The order $\lhd$ can be described as follows. Assume that Δ lies to the left of the e, directed from endpoint u to endpoint v; direct all edges in P_e^* from pixel(u) to pixel(v). If facets $f_0, f_1 \in F_\Delta$ have bounding edges $e_0^*, e_1^* \in P_e^*$, then $f_0 \lhd f_1$ if e_0^* is to the left of e_1^*; all facets covering e appear at the end of the order $\lhd$, and are ordered arbitrarily among themselves.

We remark that there may be facets in F_e that are not in F_Δ for either of the two possible triangles Δ incident to e; these facets do not appear in the covering order. Such facets necessarily have two edges in P_e^*, both of which project and snap-round to e. Such edges (and edges in P_e^* not incident to any facet) are important for the definition of $\sigma(f)$ and are discussed in section 6.3 below.

For a set $S \subset \mathbb{R}^3$, let $V(S)$ be all points on all vertical lines through S. Let f_A be a facet of P, e an edge of T with endpoints u and v, and A an edge over e. Edge A *approximates* f_A *at* e if $d(A_u, f_A \cap V(R_e)) < \kappa$ and $d(A_v, f_A \cap V(R_e)) < \kappa$. Clearly, if A approximates f_A at e, then by proposition 3.1, $d(A, f_A) < \kappa$. Also, if $e^* \in P_e^*$ is a boundary edge of face f, then $d(\tau(e^*), e^*) < \kappa$ by Lemma 4.3, and $\tau(e^*)$ approximates f at e.

Lemma 5.1 *(Snapping Lemma) Let edge e bound triangle Δ of T. Suppose $f_A, f_B, f_C \in F_\Delta$ with $f_A \succeq f_C \succeq f_B$, A, B, C are edges over e approximating f_A, f_B, f_C, respectively, $A \succeq B$, and f_C covers f_A and f_B. Then* snap$(C, [A, B])$ *also approximates f_C at e.*

Proof: We claim $\max(B, C)$ approximates f_C at e; a similar result holds for min, from which the lemma follows. Let u be an endpoint of e. We show $d(\max(B_u, C_u), f_C \cap V(R_e)) < \kappa$. If $C_u \succeq B_u$, there is nothing to prove, so suppose $C_u \prec B_u$.

Let T_B and T_C be the cubes of sidelength 2κ centered at B_u and C_u, respectively, and $T = V(T_B)$ (clearly also $T = V(T_C)$). See figure 7.

Since B approximates f_B, there is a point $b \in f_B \cap V(R_e) \cap T_B$. Since $f_C \succeq f_B$ and f_C covers f_B at e, there is a point $c \in f_C$ with $c \succeq b$; clearly $c \in T$. Since C approximates f_C, there is a point

$c' \in f_C \cap V(R_e) \cap T_C$. Since $f_C \cap V(R_e) \cap T$ is path-connected, there is a path in $f_C \cap V(R_e) \cap T$ from c to c'. Since c is above the bottom facet of T_B, c' is below the top facet of T_C, and $T_B \succeq T_C$, some point of the path meets T_B. Hence $d(B_u, f_C) < \kappa$. □

5.2 Default edges.

Let e be an edge of T_f with endpoints u and v and with some triangle incident. We define the *default lifting of edge e for facet f*, $c_f(e)$, which is to be used in the absence of other constraints. If e is a boundary edge of T_f, then there is a unique edge $e^* \in P_e^*$ bounding facet f, and we simply define $c_f(e) = \tau(e^*)$.

The definition of $c_f(e)$ is more complex if e is an interior edge of T_f. Clearly f covers e and no edge in P_e^* meets f.

Define $\mathrm{low}_f(e)$ to be the edge $\hat{u}\hat{v}$, where $\hat{u}$ is the center of the lowest voxel X in column(u) so that $X \cap f \cap V(R_e)$ is not empty, and similarly for $\hat{v}$. A pair of distinct edges (a^*, b^*) in P_e^* is a *bracketing pair* if $a^* \succ f$ and $f \succ b^*$, $\tau(a^*) \succeq \tau(b^*)$ and no edge $\tau(d^*)$, $d^* \in P_e^*$, lies between $\tau(a^*)$ and $\tau(b^*)$ (possibly $\tau(a^*) = \tau(b^*)$). The existence of a bracketing pair can be seen by indexing the edges of $P_e^* = \{e_0^*, \ldots, e_k^*\}$ so that $\tau(e_0^*) \succeq \tau(e_1^*) \succeq \ldots \succeq \tau(e_k^*)$. Either $f \succ e_0^*$, and the pair $(\top, e_0^*)$ suffices (with the definition $\tau(\top) = \top$); or $e_k^* \succ f$, and the pair $(e_k^*, \bot)$ suffices; or there is i so that an $e_i \succ f$ and $f \succ e_{i+1}$, and the pair (e_{i+1}, e_i) suffices. It is possible that there are several bracketing pairs. Define

$$c_f(e) = \mathrm{snap}(\mathrm{low}_f(e), [\tau(a^*), \tau(b^*)]),$$

where (a^*, b^*) is a bracketing pair chosen so that $(\tau(a^*), \tau(b^*))$ is minimal in $\prec$ among bracketing pairs. Set $C_e = \{c_f(e) : f \in F_\Delta\}$.

Lemma 5.2 *Let f be a facet of P and e an edge of T_f.*

1. *$c_f(e)$ approximates f at e.*
2. *If f, f' cover e and $f \preceq f'$, then $c_f(e) \preceq c_{f'}(e)$.*
3. *$\tau(P_e^*) \cup C_e$ is noncrossing.*

Proof:

1. If e is a bounding edge of T_f, then the claim is immediate. Otherwise e is an interior edge of T_f and

$$c_f(e) = \mathrm{snap}(\mathrm{low}_f(e), [\tau(a^*), \tau(b^*)]),$$

for some bracketing pair (a^*, b^*). Let a^* and b^* be incident to faces f_a and f_b, respectively. It is easy to check that $\mathrm{low}_f(e)$ approximates f. Clearly $\tau(a^*)$ approximates f_a, $\tau(b^*)$ approximates f_b, and f covers f_a and f_b (since e is not a bounding edge of T_f). Part (1) is thus immediate from lemma 5.1.

2. We have $R_e \subseteq \pi_{xy}(f)$, $R_e \subseteq \pi_{xy}(f')$, and $f \prec f'$, so $\mathrm{low}_f(e) \preceq \mathrm{low}_{f'}(e)$. Let a_f^*, b_f^* and $a_{f'}^*, b_{f'}^*$ be bracketing pairs for f and f', respectively. We claim $\tau(a_f^*) \preceq \tau(a_{f'}^*)$ and $\tau(b_f^*) \preceq \tau(b_{f'}^*)$, from which $c_f(e) \preceq c_{f'}(e)$ follows easily. Clearly $\tau(a_f^*) \succeq \tau(b_f^*)$. It cannot be that $\tau(b_f^*) \succ \tau(a_{f'}^*)$, for

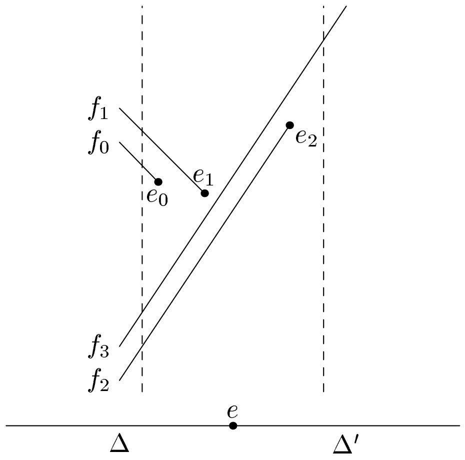

Figure 8: Definition of $l_{f\Delta}(e)$, side view. Δ is incident to the left and Δ' to the right. Vertical dotted lines outline area that projects and snap-rounds to e.

$a^*_{f'} \succ f' \succeq f$ and f would have a bracketing pair below (a^*_f, b^*_f), contradicting minimality. No edge of $\tau(P^*_e)$ lies between $\tau(a^*_f)$ and $\tau(b^*_f)$, so it must be that $\tau(a^*_{f'}) \succeq \tau(a^*_f)$. Similarly $\tau(b^*_{f'}) \succeq \tau(b^*_f)$.
3. By the definition of "snap", no edge $c_f(e)$ crosses an edge of $\tau(P^*_e)$. Also clearly, if f and f' have distinct bracketing pairs, then $c_f(e)$ and $c_{f'}(e)$ do not cross. If f and f' have the same bracketing pair, then $c_f(e)$ and $c_{f'}(e)$ do not cross because $\mathrm{low}_f(e)$ and $\mathrm{low}_{f'}(e)$ do not cross. $\Box$

5.3 Lifting triangle edges.

Let e be an edge of triangle Δ of T. For facets $f \in F_\Delta$ in the order $\lhd$, simultaneously and inductively define $a_{f\Delta}(e)$ (the *constraint from above*), $b_{f\Delta}(e)$ (the *constraint from below*) and $l_{f\Delta}(e)$ (the *lifting of edge e of Δ in f*), as follows:

$$\begin{aligned} a_{f\Delta}(e) &= \min\{l_{f'\Delta}(e) : f' \lhd f \text{ and } f' \succ f\} \\ b_{f\Delta}(e) &= \max\{l_{f'\Delta}(e) : f' \lhd f \text{ and } f' \prec f\} \\ l_{f\Delta}(e) &= \mathrm{snap}(c_f(e), [a_{f\Delta}(e), b_{f\Delta}(e)]). \end{aligned}$$

We have $a_{f\Delta}(e) \succeq b_{f\Delta}(e)$ by lemma 5.3(1) below.

The definition is illustrated schematically in figure 8. For this configuration, we have

$$\begin{aligned} l_{f_0\Delta}(e) &= \mathrm{snap}(\tau(e_0^*), [\top, \bot]) = \tau(e_0^*) \\ l_{f_1\Delta}(e) &= \mathrm{snap}(\tau(e_1^*), [\top, l_{f_0\Delta}(e)] = \tau(e_0^*) \\ l_{f_2\Delta}(e) &= \mathrm{snap}(\tau(e_2^*), [l_{f_0\Delta}(e), \bot] = \tau(e_0^*) \\ l_{f_3\Delta}(e) &= \mathrm{snap}(c_{f_3}(e), [l_{f_0\Delta}(e), l_{f_2\Delta}(e)] = \tau(e_0^*). \end{aligned}$$

$F_{\Delta'}$ contains only f_3, and $l_{f_3\Delta'}(e) = c_{f_3}(e)$.

Lemma 5.3 *Let e be an edge of triangle Δ of T with edge e, and let $f, f' \in F_\Delta$.*

1. $a_{f\Delta}(e) \succeq b_{f\Delta}(e)$.
2. *If $f \preceq f'$, then $l_{f\Delta}(e) \preceq l_{f'\Delta}(e)$.*
3. *$l_{f\Delta}(e)$ approximates f at e.*
4. *$l_{f\Delta}(e) \in \tau(P_e^*) \cup C_e$.*

Proof: 1. and 2. We prove both simultaneously by induction on $\lhd$. If $a_{f\Delta}(e) = \top$ or $b_{f\Delta}(e) = \bot$, $a_{f\Delta}(e) \succeq b_{f\Delta}(e)$ is immediate. Otherwise $a_{f\Delta}(e) = l_{f_0\Delta}(e)$ and $b_{f\Delta}(e) = l_{f_1\Delta}(e)$ for some facets $f_0 \succeq f \succeq f_1$, so by induction hypothesis $a_{f\Delta}(e) \succeq b_{f\Delta}(e)$. For (2), suppose $f \preceq f'$; without loss of generality assume $f' \lhd f$. Then by definition $l_{f'\Delta}(e) \succeq a_{f\Delta}(e) \succeq l_{f\Delta}(e)$.

3. Since $l_{f\Delta}(e)$ is defined in the order $\lhd$, the claim follows from an easy induction using lemma 5.1.

4. By lemma 5.2, $\tau(P_e^*) \cup C_e$ is noncrossing, so the "snap" in the definition of $l_{f\Delta}(e)$ results in an element of $\tau(P_e^*) \cup C_e$. □

For e an edge of T, define $L_e = C_e \cup \{l_{f\Delta}(e) : \Delta \text{ is incident to } e \text{ and } f \in P_\Delta\}$. L_e is all of the edges that have been defined over e.

Corollary 5.4 *L_e is noncrossing.*

Proof: Lemmas 5.3(4) and 5.2(3), using the fact that the construction of C_e does not depend upon the choice of triangle incident to e. □

6 The subdivision Q

In this section we define the subdivision Q and the embedding σ of P into Q required by theorem 2.1. This section also contains the definitions of l_f for the vertices, edges, and triangles of T.

6.1 Vertices and edges

Let v^* be a vertex of P^*. Define $\sigma(v^*) = \rho(v^*)$ (recall $\rho(v^*)$ is the center of the voxel containing v^*).

Let v be a vertex of T. The *vertical carrier* $VC(v)$ is the vertical chain of edges through $\rho(P_v^*)$, i.e. all edges connecting two vertices of $\rho(P_v^*)$ that are adjacent in vertical order.

Let $e^* \in P_e^*$, where edge e in T has endpoints u and v. Define $\sigma(e^*)$ to be the subdivision consisting of $\tau(e^*)$, the subchain of $VC(u)$ connecting $\tau(e^*)_u$ to $\rho(e^*)_u$ and the subchain of $VC(v)$ connecting $\tau(e^*)_v$ to $\rho(e^*)_v$. Extend σ to edges e of P:

$$\sigma(e) = \bigcup_{e^* \in P^*,\, e^* \subseteq e} \sigma(e^*)$$

Clearly $\sigma(e)$ is a subdivision.

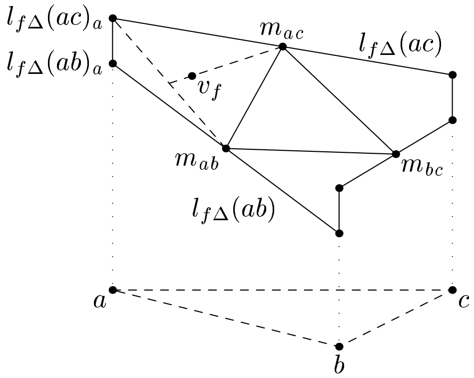

Figure 9: Definition of v_f for triangle Δ of T_f with vertices a, b, c.

Lemma 6.1 *If w, w' are vertices or edges of P and $w \preceq w'$, then $\sigma(w) \preceq \sigma(w')$.*

Proof: The claim is immediate for two vertices. Suppose w is a vertex and w' an edge; the symmetric case is similar. Then $\sigma(w')$ contains a vertical chain of edges from the center of the first voxel in column(w) met by w' to the center of the last such voxel. Since $w \prec w'$, $\rho(w)$ is below or on the chain, and $\sigma(w) \preceq \sigma(w')$. The case of two edges is similar. □

Let e be an edge of T with endpoints u and v. Split each edge in $L(e)$ at its midpoint. Split each edge at its midpoint. These edges together with $VC(u)$ and $VC(v)$ form a planar graph (in the plane through $VC(u)$ and $VC(u)$). The *vertical carrier* $VC(e)$ is an arbitrary triangulation of this graph.

6.2 Triangles

Let Δ be a triangle of T_f with vertices a, b, c. Consider the edges $l_{f\Delta}(ab)$, $l_{f\Delta}(ac)$, $l_{f\Delta}(bc)$. There is no guarantee that these edges are pairwise incident (of course both $l_{f\Delta}(ab)$ and $l_{f\Delta}(ac)$ are incident to vertices over a, and similarly for the other pairs). We form a (three-dimensional) polygon from $l_{f\Delta}(ab)$, $l_{f\Delta}(ac)$, $l_{f\Delta}(bc)$ by adding the vertical subchain of $L(a)$ connecting $l_{f\Delta}(ab)_a$ to $l_{f\Delta}(ac)_a$ (if they are not equal) and similarly for the b and c endpoints. The *lifting of Δ for facet f*, $l_f(\Delta)$, is a triangulation of this polygon, described as follows.

Split edges $l_{f\Delta}(ab)$, $l_{f\Delta}(ac)$, $l_{f\Delta}(bc)$ at their respective midpoints m_{ab}, m_{ac}, m_{bc}, and add the three edges connecting midpoints. This forms a central triangle $m_{ab}m_{ac}m_{bc}$ and three polygons, where for example the *a-polygon (of f)* consists of edge $m_{ab}m_{ac}$, the two subedges of $l_{f\Delta}(ab)$ and $l_{f\Delta}(ac)$ with endpoints over a, and possibly a vertical chain over a. See figure 9.

For points $p, q \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, let $\alpha[p, q]$ be the point $(1-\alpha)p + \alpha q$, i.e. the point a fraction α of the way from p to q.

The *a-index* of f is the number of distinct pairs $(l_{f'\Delta}(ab), l_{f'\Delta}(ac))$, where $f' \succeq f$. Let $\alpha_f = i/2^{\lceil \log_2 n \rceil}$, where i is the a-index of f; clearly $0 < \alpha_f < 1$. First assume $l_{f\Delta}(ac)_a \succeq l_{f\Delta}(ab)_a$. Set $v_f = \alpha_f[\ \frac{1}{2}[l_{f\Delta}(ac)_a, m_{ab}], m_{ac}]$. See figure 9. Triangulate the a-polygon of f with v_f, i.e. connect v_f to m_{ac}, m_{ab}, and any vertex on the chain from $l_{f\Delta}(ab)_a$ to $l_{f\Delta}(ac)_a$. If $l_{f\Delta}(ab)_a \succ l_{f\Delta}(ac)_a$, the construction is similar, with m_{ab} and m_{ac} interchanged and $l_{f\Delta}(ab)_a$ substituting for $l_{f\Delta}(ac)_a$. The other two polygons are triangulated in a similar fashion.

Lemma 6.2 *Let Δ be a triangle of T and $f, f' \in F_\Delta$.*

1. $d(l_f(\Delta), f) < \kappa$.
2. *If $f \preceq f'$, then $l_f(\Delta) \preceq l_{f'}(\Delta)$.*
3. *Every vertex coordinate of $l_f(\Delta)$ is an integral multiple of $1/2^{\lceil \log_2 n \rceil + 2}$.*
4. *$l_f(\Delta)$ has $O(n)$ cells.*

Proof:

1. Let Δ have vertices a, b, c. Every vertex of $l_f(\Delta)$ is within the convex hull of $\{l_{f\Delta}(ab), l_{f\Delta}(ac), l_{f\Delta}(bc)\}$. The claim follows using lemma 5.3(1) and proposition 3.1.

2. We can assume that $l_{f\Delta}(ac)_a \succeq l_{f\Delta}(ab)_a$. Using lemma 5.3(2), we must have

$$l_{f'\Delta}(ac)_a \succeq l_{f\Delta}(ac)_a \text{ and } l_{f'\Delta}(ab)_a \succeq l_{f\Delta}(ab)_a.$$

If $l_{f'\Delta}(ab)_a \succ l_{f\Delta}(ac)_a$, then the result is immediate, since the convex hull of $\{l_{f\Delta}(ab), l_{f\Delta}(ac)\}$ and the convex hull of $\{l_{f'\Delta}(ab), l_{f'\Delta}(ac)\}$ have disjoint interiors. Hence we can assume that

$$l_{f'\Delta}(ac)_a \succeq l_{f\Delta}(ac)_a \succeq l_{f'\Delta}(ab)_a \succeq l_{f\Delta}(ab)_a.$$

Let i_f and $i_{f'}$ be the a-indices of f and f' respectively. If $i_f = i_{f'}$, then the a-polygons for f and f' are identical. Otherwise, $i_f > i_{f'}$ since $f \preceq f'$. Let s be the edge connecting $v_{f'}$ to the midpoint of $l_{f'\Delta}(ac)$; clearly we have $\pi_{xy}(v_f) \in \pi_{xy}(s)$ since $\alpha_f > \alpha_{f'}$. Furthermore we have $v_f \prec s$, since $l_{f\Delta}(ab) \preceq l_{f'\Delta}(ab)$ and $l_{f\Delta}(ac) \preceq l_{f'\Delta}(ac)$ with inequality holding in at least one case. $l_f(\Delta) \preceq l_{f'}(\Delta)$ follows easily.

3., 4. Immediate. □

6.3 Vertical ordering

It is tempting to define $\sigma(f) = \bigcup_{\Delta \in T_f} l_f(\Delta)$. By lemma 6.2(2), this definition would preserve or collapse vertical order (in the sense of theorem 2.1) among lifted triangles. However, order would not necessarily be preserved between lifted triangles and rounded edges or vertices. To see why, let e be an edge of T_f with two triangles Δ and Δ' incident. It is possible that there is an isolated edge $e^* \in P_e^*$ with no facet of P incident (or similarly an edge e^* of facet $f \in P_e \setminus (P_\Delta \cup P_{\Delta'})$, i.e. f has two edges projecting and snap-rounding to e). It is furthermore conceivable that $e^* \prec f$ but that both $l_{f\Delta}(e) \prec \tau(e^*)$ and $l_{f\Delta'}(e) \prec \tau(e^*)$. With the tempting definition above, the vertical order between e^* and f would not be preserved by rounding. The solution, given below, is to specially define $l_f(e)$ as a (triangulated) vertical polygon, and include $l_f(e)$ in $\sigma(f)$. Similarly, for a vertex $v \in T$, $l_f(v)$ is defined as a vertical chain. We remark that in consequence $\sigma(f)$ may not be a 2-manifold; it may include vertical chains and polygons over vertices and edges of T.

For a facet f, let $E_f(v)$ be all endpoints of edges $\{l_{f\Delta}(e) : e, \Delta \in T\}$ over v (clearly edge e must be incident to v and Δ). Define

$$\begin{aligned} a_f(v) &= \min\Big(\{\sigma(v^*) : v^* \in P_v^* \text{ and } v^* \succeq f\} \cup E_f(v)\Big) \\ b_f(v) &= \max\Big(\{\sigma(v^*) : v^* \in P_v^* \text{ and } v^* \preceq f\} \cup E_f(v)\Big). \end{aligned}$$

Informally, $a_f(v)$ is the lowest snap-rounding over v of a vertex on or above f; similarly $B_f(v)$ is the highest snap-rounding over v of a vertex on or below f. Easily $a_f(v) \preceq b_f(v)$. The *lifting of vertex v for facet f, $l_f(v)$*, is the subchain of $VC(v)$ connecting $a_f(v)$ and $b_f(v)$.

For a facet f and an edge e of T_f, let $E_f(e) = \{l_{f\Delta}(e) : \Delta \text{ incident to } e \text{ in } T_f\}$. Clearly, there are are zero, one, or two edges in $E_f(e)$ as there are zero, one, or two triangles incident to e in T_f. Define

$$\begin{aligned} a_f(e) &= \min\Big(\{\tau(e^*) : e^* \in P_e^* \text{ and } e^* \succeq f\} \cup E_f(e)\Big) \\ b_f(e) &= \max\Big(\{\tau(e^*) : e^* \in P_e^* \text{ and } e^* \preceq f\} \cup E_f(e)\Big). \end{aligned}$$

Notice that if $E_f(e)$ is empty, then there must be some edge of $e^* \in P_e^*$ incident to f, so $a_f(e)$ and $b_f(e)$ are distinct from $\perp$ and $\top$, respectively. The *lifting of edge e for facet f, $l_f(e)$*, is all edges and vertices w of $VC(e)$ satisfying $b_f(e) \succeq w$ and $w \succeq a_f(e)$.

Lemma 6.3 *Suppose w is a vertex or edge of T, $w^* \in P_w^*$, and f is a facet of P. Then $w^* \preceq f$ implies $\sigma(w^*) \preceq l_f(w)$ and $w^* \succeq f$ implies $\sigma(w^*) \succeq l_f(w)$.*

Proof: By construction. □

Lemma 6.4 *Let f be a facet of P and w a vertex or edge of T_f. Then $d(l_f(w), f) \leq \kappa$.*

Proof: Similar to the proof of lemma 5.1. □

For each facet f of P, define

$$\sigma(f) = \bigcup_{w \in T_f} l_f(w),$$

where w varies over vertices, edges, and triangles. It is easy to check that $\sigma(f)$ is a subdivision.

Lemma 6.5 *If f, f' are cells of P and $f \prec f'$, then $\sigma(f) \preceq \sigma(f')$.*

Proof: The lemma follows from lemmas 6.1 and 6.3 if one of f and f' is a vertex or edge. So suppose both are facets. For each triangle Δ in both T_f and $T_{f'}$, $l_f(\Delta) \preceq l_{f'}(\Delta)$ by lemma 6.2. Suppose e is an edge in both T_f and $T_{f'}$. If there is a triangle Δ in both T_f and $T_{f'}$ incident to e, lemma 6.2 again implies $l_f(e) \preceq l_{f'}(e)$. Otherwise, up to symmetry, there is an edge $e^* \in P_e^*$ bounding f with $e^* \preceq f'$, so by lemma 6.3, $\sigma(e^*) \preceq l_{f'}(e)$. Since $e^* \preceq f$, $\sigma(e^*) \subseteq l_f(e)$, and $l_f(e) \preceq l_{f'}(e)$. A similar argument shows that if v is a vertex in both T_f and $T_{f'}$, then $l_f(v) \preceq l_{f'}(v)$. Hence $\sigma(f) \preceq \sigma(f')$. □

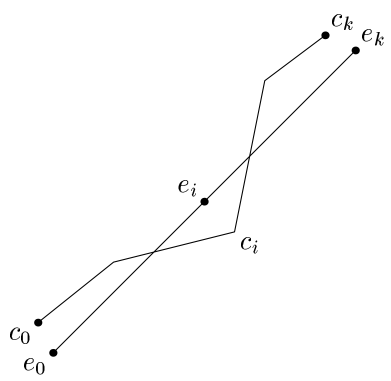

Figure 10: C is the chain c_0c_1, …, $c_{k-1}c_k$; edge E is e_0e_k.

6.4 The subdivision Q

Let

$$Q \;=\; \bigcup_f \sigma(f),$$

where f varies over all facets of P. It is easy to check that Q is a subdivision and that σ is an embedding of P into Q.

Lemma 6.6 *Q has $O(n^4)$ cells and can be computed in time $O(n^4)$.*

Proof: For each facet f of P, T_f has $O(n^2)$ triangles Δ. By lemma 6.2, $l_f(\Delta)$ has $O(n)$ cells. Hence $\sigma(f)$ has $O(n^3)$ cells, for a total of $O(n^4)$ over all facets of f. Q can easily be computed in the same time. □

7 Hausdorff distance

It is immediate from lemmas 6.2 and 6.4 that $d(\sigma(f), f) \le \kappa$. In this section we show that $d(f, \sigma(f)) \le \kappa$, implying $d_H(\sigma(f), f) \le \kappa$ and completing the proof of theorem 2.1. This part of the proof has a topological flavor.

To illustrate the proof, we first give a one-dimensional analogue. Suppose we have a line segment $E = e_0e_k$ and a polygonal chain $C = c_0c_1, c_1c_2, \ldots, c_{k-1}c_k$ satisfying

1. $d(c_i, E) < \kappa$, and
2. $d(e_0, c_0) < \kappa$ and $d(e_k, c_k) < \kappa$;

See figure 10. Note that (1) immediately implies $d(C, E) < \kappa$ using proposition 3.1; we wish to establish $d(E, C) < \kappa$ (which is clearly false without condition (2)).

The proof is sketched as follows, using the same terminology as the two-dimensional case below. For $i = 1, \ldots, k-1$, choose e_i as the point of E closest to c_i, then $d(c_i, e_i) < \kappa$ for all $i = 0, \ldots, k$.

Let A be a chain of k (abstract) edges, $A = a_0a_1, \dots, a_{k-1}a_k$. We can view C and E as embeddings ϕ_C and ϕ_E of A, i.e. $\phi_C(a_i) = c_i$ and $\phi_E(a_i) = e_i$. Then we have easily (compare lemma 7.3)

(a) $\phi_E(A) \subset E$ and $\phi_E(A)$ *covers* E, i.e. ϕ_E maps the endpoints of A to the endpoints of E.

(b) Embeddings ϕ_E and ϕ_C are *close*, i.e. $d(\phi_C(a_i), \phi_E(a_i)) < \kappa$ for $i = 0, \dots, k$.

Item (a) implies that $E = \phi_E(A)$ (compare lemma 7.1) while (b) implies that $d_H(\phi_E(A), \phi_C(A)) < \kappa$ (compare lemma 7.2); these two assertions together yield the desired conclusion.

We now return to the two-dimensional case. An *abstract triangle*, Δabc, is a cyclically ordered set of distinct *abstract vertices* a, b, c (so $\Delta abc = \Delta bca = \Delta cab \neq \Delta acb$); Δabc has *directed edges* ab, bc, and ca. An *abstract triangulation* A is a set of abstract triangles so that for each directed edge ab, there is a unique triangle with directed edge ba. An abstract triangulation A has *boundary* $\partial A = \{a_0a_1, a_1a_2, \dots, a_ka_0\}$ if there is a distinguished vertex i (the *point at infinity*) so that Δia_1a_0, …, Δia_ka_{k-1}, Δia_0a_k are exactly the triangles that have i as a vertex. An *embedding* ϕ of an abstract triangulation is a mapping from vertices (except the point at infinity) into $\mathbb{R}^3$. Embedding ϕ extends to edges, triangles, and all of A: $\phi(ab) = \phi(a)\phi(b)$, $\phi(\Delta abc)$ is the convex hull of $\{\phi(a), \phi(b), \phi(c)\}$, and $\phi(A) = \bigcup_{\Delta \in A} \phi(\Delta)$. Embedding ϕ may map two vertices to the same point of $\mathbb{R}^3$, cause two triangles to intersect, etc.

Let ϕ be an embedding of triangulation A with boundary ∂A and let f be a facet of P; ϕ *covers* f if $\phi(v) \in f$ for all vertices $v \in A$; $\phi(\partial A) = \partial f$; and for distinct edges e, e' of ∂A, $\phi(e)$ and $\phi(e')$ have disjoint interiors.

Lemma 7.1 *Let ϕ be an embedding of A. If ϕ covers f, then $f = \phi(A)$.*

Proof: Clearly $\phi(A) \subseteq f$. For the converse, let $p \in f$. Choose a directed line l through p in the plane of f so that l avoids all vertices of $\phi(A)$. Choose an arbitrary orientation of the plane through f. Consider the directed graph whose nodes are the triangles Δ of A so that $\phi(\Delta) \cap l \neq \emptyset$ and whose arcs are directed from a triangle Δabc to a triangle Δacd if a lies to the left of l and c to its right. Clearly each triangle has indegree at most one and outdegree at most one. There is a unique edge in ∂A that contains the first point of $l \cap f$; hence there is a unique triangle Δ_0 of indegree 0. Similarly, there is a unique triangle Δ_k of outdegree 0. Hence there is a path of triangles $\Delta_0, \Delta_1, \dots, \Delta_k$. Consider the edges $e_i = \phi(\Delta_i) \cap l$. Consecutive edges share endpoints, so the union of the edges is $l \cap f$. Hence $p \in e_i$, some i, and $p \in \phi(\Delta_i) \subseteq f$. □

An embedding $\hat{\phi}$ of A is *close* to embedding ϕ if for all vertices v of A, $d(\hat{\phi}(v), \phi(v)) < \kappa$.

Lemma 7.2 *If embeddings $\phi, \hat{\phi}$ of A are close, then $d_H(\phi(A), \hat{\phi}(A)) < \kappa$.*

Proof: Immediate using proposition 3.1. □

Lemma 7.3 *For each facet f of P, there is an abstract triangulation A and close embeddings $\phi, \hat{\phi}$ so that $\hat{\phi}$ covers f and $\phi(A) \subseteq \sigma(f)$.*

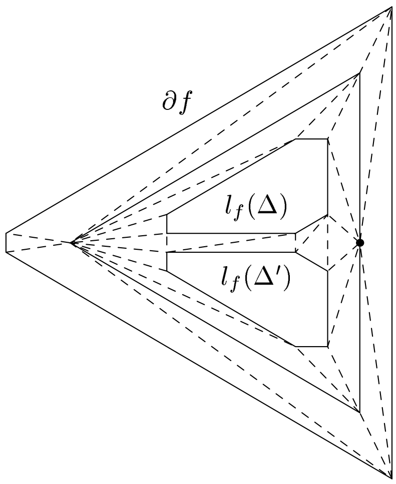

Figure 11: Abstract triangulation A. Outer solid cycle is copy of ∂f; middle solid cycle is copy of $\sigma(\partial(f)$; inner polygons are copies of $l_f(\Delta)$ and $l_f(\Delta')$ (internal edges not shown). Dotted edges are connecting triangulation edges.

Proof: We first assume that T_f has at least one triangle and that every edge of T_f is incident to a triangle. A is obtained by pasting together various subtriangulations, using T_f as a guide. For each triangle Δ in T_f, A has an abstract copy of $l_f(\Delta)$, i.e. a set of abstract triangles with the same incidence structure as $l_f(\Delta)$; ϕ maps each abstract vertex to the corresponding vertex of $l_f(\Delta)$. A contains an abstract copy of $\sigma(\partial f)$, i.e. an abstract cycle formed from a copy of $\sigma(e)$ for each edge e in ∂f; ϕ maps each abstract vertex to the corresponding vertex of $\sigma(\partial f)$. The boundary of A is formed by an abstract copy of ∂f; ϕ maps each abstract vertex to the image under σ of the corresponding vertex of ∂f. (Each edge in the boundary of A forms a triangle with the point at infinity.)

The abstract copies are connected together as follows. For each internal edge e between two triangles Δ and Δ', the abstract copies of $l_{f\Delta}(e)$ and $l_{f\Delta'}(e)$ are connected with intermediate abstract triangles (see figure 11). Similarly, e is a boundary edge of T_f and is incident to a triangle Δ, the abstract copies of $\sigma(e)$ and $l_{f\Delta}(e)$ are connected by intermediate abstract triangles. For each vertex v of T_f, the vertices of A that are abstract copies of v have been connected to form a cycle; this cycle is now triangulated. Finally, the cycles formed by abstract copies of ∂f and $\sigma(\partial(f))$ are connected: each copy of a vertex $v \in \partial f$ is connected by an edge to the copy of $\sigma(v) \in \sigma(\partial(f))$, and each copy of an edge $e \in \partial f$ is connected by intermediate triangles to is connected to the copy of $\sigma(e) \in \sigma(\partial(f))$.

Clearly we have $\phi(A) \subseteq \sigma(f)$. Define $\hat{\phi}$ on ∂A by mapping the copy of a vertex $v \in \partial f$ to v; define $\hat{\phi}$ elsewhere by mapping abstract vertex $u \in A$ to the closest point on f to $\phi(u)$. Clearly $\hat{\phi}$ covers f and is close to ϕ.

If T_f has an edge e without incident triangles, the approach is similar, using an abstract copy of $l_f(e)$. The case that T_f consists of a single vertex can be handled trivially. □

Corollary 7.4 *If f is a facet of P, then $d_H(f, \sigma(f)) < \kappa$.*

Proof: Immediate from lemmas 7.1, 7.2, and 7.3. □

8 Discussion

It may be possible to improve the worst-case bounds given in theorem 2.1. For example, the $O(n^4)$ bound on the size of Q could be an artifact of vertical projection; perhaps an $O(n^3)$ bound could be obtained by using different projection directions in different places, each tuned to the local configuration. Obtaining a worst-case bound below $O(n^3)$ seems very challenging. It would be desirable to remove the extra $\lceil \log_2 n \rceil + 2$ bits needed for vertex coordinates; again, this may be an artifact of vertical projection. Finally, it would be desirable to guarantee that the rounded image of a facet at least was locally a 2-manifold.

A programmer would probably prefer a simple rounding algorithm, even at the expense of degraded worst-case bounds, as long as the typical-case bounds are reasonable. One reason that the rounding algorithm is complicated is the need to avoid edge crossings. Milenkovic [17] suggests rounding existing vertices to integer coordinates. If two rounded edges cross, then a vertex of intersection is added, with coordinates computed exactly. This would require a constant-factor increase in the bit-length of some vertex coordinates, and hence of some predicate evaluations. However, the maximum required bit-length is still bounded, and perhaps the increased-length calculations are relatively infrequent. Perhaps this approach can lead to a practical rounding algorithm.

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A Proof of lemma 4.2

Proof: Let e have endpoints u and u', e^* have endpoints E and E', and d^* have endpoints D and D', where $D, E \in \text{column}(u)$ and $D', E' \in \text{column}(u')$. By the definition of P^*, d^* and e^* each intersect only a single voxel of column(u) and a single voxel of column(u'). By clipping d^* and e^* slightly, we can assume that D and E lie on a bounding facet of column(u) and D' and E' lie on a bounding facet of column(u').

We write e.g. D_x for the x-coordinate of D. Without loss of generality we can assume $D_x < D'_x$, $D_y < D'_y$, $D_z < D'_z$, and since $\rho(d^*)$ and $\rho(e^*)$ cross, we can assume that $E_z > D_z$ and $E'_z < D'_z$. It cannot be that $u_y = u'_y$, for then D, D', E, E' would lie on facets of column(u) and column(u') parallel to the xz-plane, and $\pi_{xz}(d^*)$ and $\pi_{xz}(e^*)$ would cross, which is impossible by

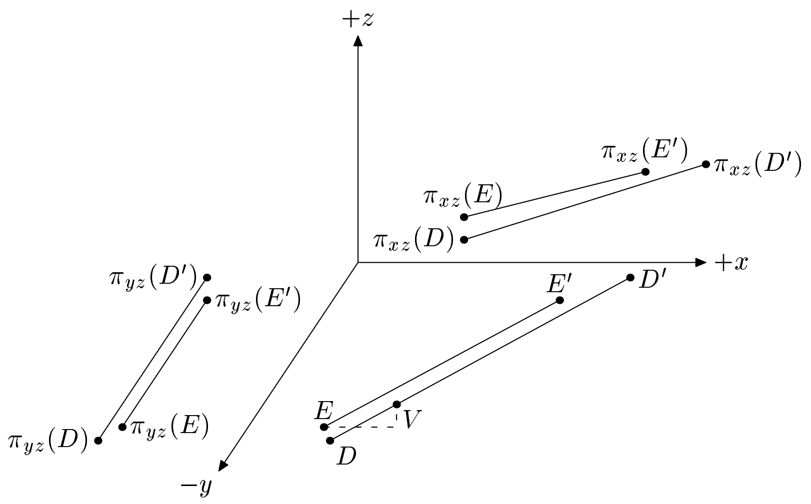

Figure 12: Case (1) of the proof of lemma 4.2.

the construction of P^*. Similarly $u_x \neq u'_x$, and we can assume that $u_x < u'_x$ and $u_y < u'_y$. Thus D and E must either lie on the xz-*facet* of column(u), that is, the facet of column(u) parallel to the xz-plane on the $+y$ side of column(u), or the yz-facet on the $+x$ side. Similarly D' and E' lie either on the xz-facet on the $-y$ side of column(u') or the yz-facet on the $-x$ side.

The proof now splits into two rather different cases, depending on whether $E_z < E'_z$ or $E_z > E'_z$.

Case 1 ($E_z < E'_z$): We have $D_z < E_z < E'_z < D'_z$. By the construction of P^*, $\pi_{xz}(d^*)$ and $\pi_{xz}(e^*)$ do not cross. We assume $\pi_{xz}(e^*) \succeq \pi_{xz}(d^*)$; the other case is symmetric, as will be evident momentarily. See figure 12. Since $\pi_{xz}(e^*) \succeq \pi_{xz}(d^*)$ and $D'_z > E'_z$, we must have $D'_x > E'_x$. Hence D' lies on the xz-facet of column(u'). The plane through the xz-facet intersects e^* at some point with z-coordinate below D'_z (since this is true for all points of e^*), hence $\pi_{yz}(d^*) \succeq \pi_{yz}(e^*)$. By a similar argument, D lies on the yz-facet of column(u). E and E' could be on either facet of column(u) and column(u'), respectively. See also figure 2. (The case $\pi_{xz}(e^*) \preceq \pi_{xz}(d^*)$ would be symmetric, with $\pi_{yz}(d^*) \preceq \pi_{yz}(e^*)$.)

Assume that the angle α between the y-axis and the line through $\pi_{xy}(d^*)$ is at most $\pi/4$; we show $d(\rho(E), d^*) \leq \kappa$. (The case $\alpha \in [\pi/4, \pi/2]$ implies $d(\rho(E'), d^*) \leq \kappa$.) Let V be the point of d^* with the same y-coordinate as E; certainly $V_z > E_z$. We have $d(\pi_{xy}(E), \pi_{xy}(D)) < 1$ and $d(\pi_{xy}(E), \pi_{xy}(V)) < 1$ since $|E_y - D_y| < 1$ and $\alpha < \pi/4$. We also have $D_z < E_z < V_z$; let W be the point on edge DV with the same z-coordinate as E. By proposition 3.1, $d(\pi_{xy}(E), \pi_{xy}(W)) < 1$, thus $d(E, W) < 1$, $d(E, d^*) < 1$, and $d(\rho(E), d^*) < 3/2 = \kappa$.

Case 2 ($E'_z < E_z$): We cannot have $E'_x = D'_x$, else $\pi_{xy}(d^*)$ and $\pi_{xy}(e^*)$ would cross. Suppose $E'_x > D'_x$. Then we have E' on the xz-facet of column(u'). Furthermore $E_z > E'_z$ and $\pi_{xz}(e^*) \succeq \pi_{xz}(d^*)$, since $\pi_{xz}(d^*)$ and $\pi_{xz}(e^*)$ do not cross. We cannot also have D' on the xz-facet of column(u'), else $\pi_{yz}(d^*)$ and $\pi_{yz}(e^*)$ would cross. Hence $D'_y > E'_y$. See figure 13. (The case $E'_x < D'_x$ would be symmetric, leading to $D'_y < E'_y$ and D' and E' interchanging facets of column(u').)

Let $\hat{E}$ be the point on e^* with the z-coordinate D'_z and $\hat{D}$ be the point on d^* with z-coordinate

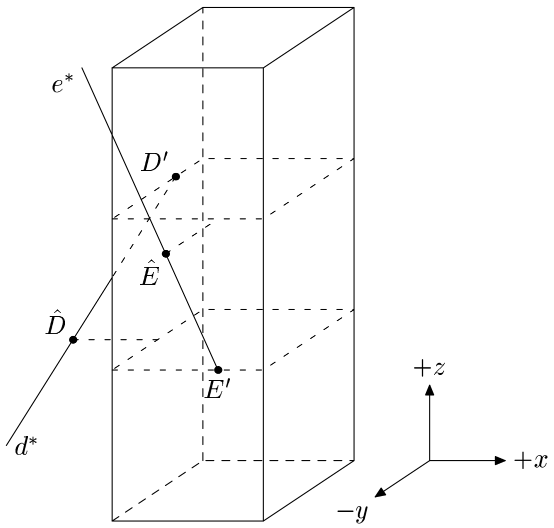

Figure 13: Case (2) of the proof of lemma 4.2. Solid outlines column(u').

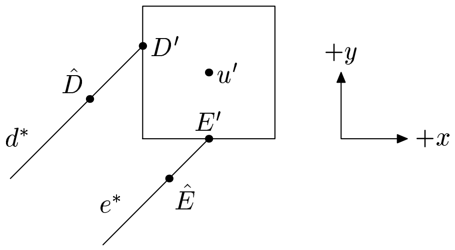

Figure 14: Case (2) of the proof of lemma 4.2. Projection onto the xy-plane. The solid square is pixel(u).

E'_z. Then $\hat{E}_x > D'_x > \hat{D}_x$ and $\hat{D}_y > E'_y > \hat{E}_y$.

The remainder of the argument occurs in the xy-plane (figure 14). We have

$$\min(\hat{D}_y - \hat{E}_y, \hat{E}_x - \hat{D}_x) < 1$$

since $d(\pi_{xy}(\hat{D}), \pi_{xy}(e^*)) < 1$ and $\pi_{xy}(e^*)$ has positive slope in the xy-plane. Hence we have either $d(\hat{D}, \mathrm{column}(u')) < 1$ or $d(\hat{E}, \mathrm{column}(u')) < 1$. It correspondingly follows that either $d(\rho(E'), d^*) \leq 3/2 = \kappa$ or $d(\rho(D'), e^*) \leq 3/2 = \kappa$. $\square$

B Symbol crossreference

Symbol	Section	Comment
$d(\cdot,\cdot)$	2	L_∞ distance (asymmetric for sets)
$d_H(\cdot,\cdot)$	2	Hausdorff distance (symmetric for sets)
$\prec, \preceq$	2	vertical order
π_{xy}	2	projection on the xy plane
A_p	2	the point of A with xy-projection p
P	2	original subdivision
Q	2	rounded subdivision
κ	2	the distance bound (3/2)
σ	2,6	embedding of P into Q
$\top, \bot$	3	symbolic sets at top and bottom of vertical order
ρ	4	(naive) 3d snap rounding
P^*	4.1	refinement of edges and vertices of P
e^*	4.1	(refined) edge of P^*
T_f	4.1	triangulation of snap-rounding of projection of f
P^*_e	4.1	edges of P^* that project and snap-round to e
P^*_v	4.1	vertices of P^* that project and snap-round to v
F_e	4.1	facets f of P that have $e \in T_f$
F_Δ	4.1	facets f of P that have $\Delta \in T_f$
τ	4.1	modified snap rounding on edges
$c_f(e)$	5.2	default choice for edges $l_{f\Delta}(e)$
L_e	5.3	all defined edges over e
VC	6.1	vertical carrier
$l_{f\Delta}(e)$	6.2	lifting of edge e of Δ for facet f
$l_f(\Delta)$	6.2	lifting of triangle Δ for facet f
$l_f(e)$	6.3	lifting of edge e for facet f
$l_f(v)$	6.3	lifting of vertex v for facet f